Gravitational lensing in terms of energy-momentum tensor and an interesting solution of Einstein equation

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Introduction: Motivations I

Friends from IATE show us their research[SR11] involving matter estimates coming from galaxy dynamics and gravitational lensing observations.

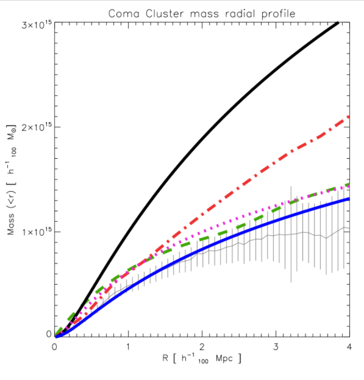


Figure: From Serra-Dominguez article.

Introduction: Motivations II

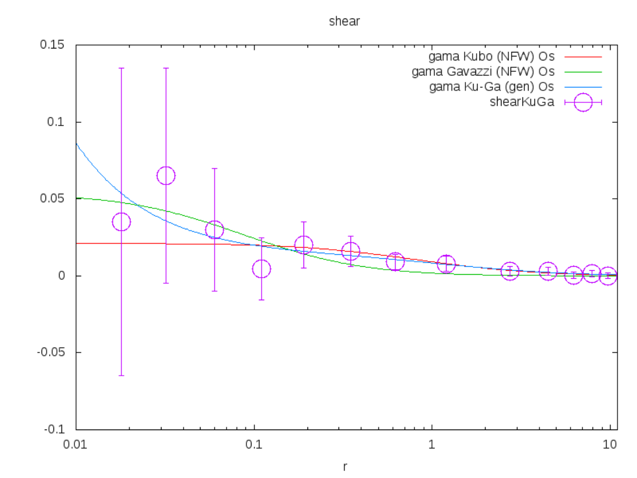


Figure: Our efforts to fit the lens data. First seven points correspond to Gavazzi[G+09] data and the last six point to Kubo[JAJ+07] data.

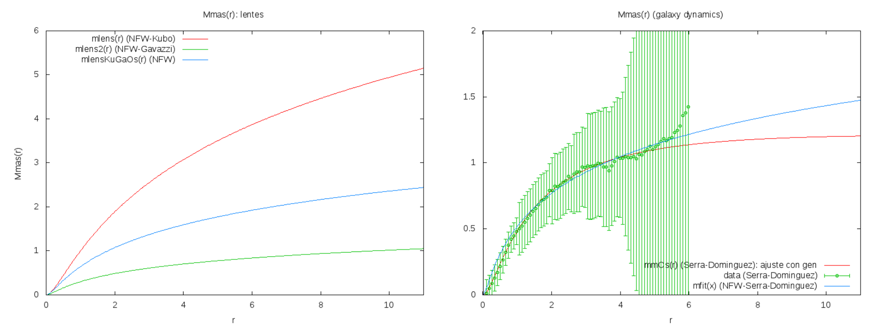


Figure: Our efforts to fit the data.

- Kubo[JAJ+07] data covers the larger radii regime.
- The NFW fit to the Kubo data seems not to agree with the matter estimates coming from galaxy dynamics studies.
- We started a systematic study of gravitational lensing.

Standard gravitational lensing notation

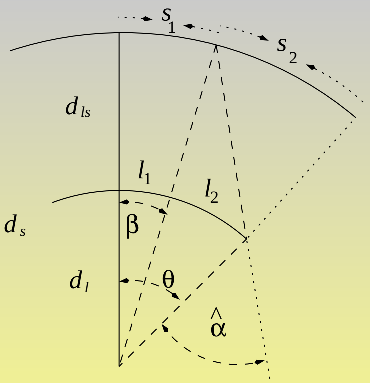


Figure: This graph shows the basic and familiar angular variables in terms of a simple flat background geometry. The letter *s* denote sources, the letter *l* denotes lens and the observer is assumed to be situated at the apex of the rays.

Introduction: Standard gravitational lensing notation II

In this framework the lens equation reads

$$\beta^a = \theta^a - \frac{d_{ls}}{d_s}\alpha^a. \tag{1}$$

The differential of this equation can be written as

$$\delta\beta^a = A^a{}_b\,\delta\theta^b, \tag{2}$$

where the matric $A^a{}_b$ is in turn expressed by

$$A^a{}_b = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}; \tag{3}$$

where the optical scalars κ, γ_1 and γ_2, are known as convergence κ and shear components $\{\gamma_1, \gamma_2\}$, and have the information of distortion of the image of the source due to the lens effects.

It is somehow striking that in most astronomical works on gravitational lensing, it is assumed that the lens scalars and deflection angle, can be obtained from a Newtonian-like potential function. These expressions although are easy to use, have some limitations:

- They neglect more general distribution of energy-momentum tensor T_{ab}, in particular they only take into account the timelike component of this tensor. In this way they severely restrict the possible candidates to dark matter that can be studied with these expressions.
- They are not expressed in terms of gauge invariant quantities.
- Since these expressions are written in terms of a potential function, it is not easily seen how different components of T_{ab} contribute in the generation of these images.
- Most of them assume from the beginning that thin lens is a good approximation.

△ We extend the work appearing in standard references on gravitational lensing[SEF92, SSE94, Wammb, Bar10] and present new expressions that do not suffer from the limitations mentioned above.

△ We present gauge invariant expressions for the optical scalars and deflection angle for some general class of matter distributions.

△ In this work we study gravitational lensing over a flat background.

Content

General equations: The geodesic deviation equation

Then a deviation vector at the source image can be expressed by

$$\varsigma^a = \varsigma \bar{m}^a + \bar{\varsigma} m^a. \tag{4}$$

Using the GHP notation one can write the geodesic deviation equation as

$$0 = \text{Þ}(\varsigma) + \varsigma\rho + \bar{\varsigma}\sigma;$$

where Þ is the well behaved derivation of type $\{1, 1\}$ in the direction of ℓ(the null geodesic vector of the congruence).
Defining $\mathcal{X}$ by

$$\mathcal{X} = \begin{pmatrix} \varsigma \\ \bar{\varsigma} \end{pmatrix}; \tag{5}$$

the equation for ς can be written as

$$\ell(\ell(\mathcal{X})) = -Q\mathcal{X}; \tag{6}$$

where Q is given by

$$Q = \begin{pmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{pmatrix}; \tag{7}$$

General equations: The geodesic deviation equation II

with

$$\Phi_{00} = -\frac{1}{2} R_{ab} \ell^a \ell^b, \tag{8}$$

and

$$\Psi_0 = C_{abcd} \ell^a m^b \ell^c m^d. \tag{9}$$

Therefore, this form of the equation only involves curvature quantities.

Approximation method for solving the geodesic deviation equation I

Approximation method for solving the geodesic deviation equation

Let us first transform to a first order differential equation. Defining $\mathcal{V}$ to be

$$\mathcal{V} \equiv \frac{d\mathcal{X}}{d\lambda}; \tag{10}$$

and

$$\mathbf{X} \equiv \begin{pmatrix} \mathcal{X} \\ \mathcal{V} \end{pmatrix}; \tag{11}$$

one obtains

$$\ell(\mathbf{X}) = \frac{d\mathbf{X}}{d\lambda} = \begin{pmatrix} \mathcal{V} \\ -Q\,\mathcal{X} \end{pmatrix} = A\,\mathbf{X}; \tag{12}$$

with

$$A \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -Q & 0 \end{pmatrix}. \tag{13}$$

Equation (12) can be re-expressed in integral form, which gives

$$\mathbf{X}(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda')\,\mathbf{X}(\lambda')\,d\lambda'. \tag{14}$$

Approximation method for solving the geodesic deviation equation II

The complete linear iteration is

$$\mathbf{X}_3(\lambda) = \begin{pmatrix} \mathbb{I} - \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\lambda'} Q''\, d\lambda''\, d\lambda' & (\lambda - \lambda_0)\mathbb{I} - \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\lambda'} (\lambda'' - \lambda_0) Q''\, d\lambda''\, d\lambda' \\ -\int_{\lambda_0}^{\lambda} Q' d\lambda' & \mathbb{I} - \int_{\lambda_0}^{\lambda} (\lambda' - \lambda_0) Q'\, d\lambda' \end{pmatrix} \mathbf{X}_0. \tag{15}$$

If the metric were flat ($Q = 0$), in order to get a deviation vector constructed from $\mathcal{X}_1$, defined as $\mathcal{X}$ evaluated at $\lambda_s = \lambda_0 + d_s$, one must choose as initial condition

$$\mathcal{V}(\lambda_0) = \frac{1}{(\lambda_s - \lambda_0)} \mathcal{X}(\lambda_s = \lambda_0 + d_s) = \frac{1}{d_s} \mathcal{X}_1. \tag{16}$$

Using a complex displacement ς of unit modulus; namely $\varsigma = e^{i\varphi}$, to represent the deviation vector, one can express the equation in the form

$$\varsigma_s(\varphi) = \left[1 - \frac{1}{d_s} \int_0^{d_s} \lambda'(d_s - \lambda') \Phi_{00}(\lambda')\, d\lambda' - \left(\frac{1}{d_s} \int_0^{d_s} \lambda'(d_s - \lambda') \Psi_0(\lambda')\, d\lambda' \right) e^{-2i\varphi} \right] e^{i\varphi}. \tag{17}$$

Optical scalar in terms of the curvature

By taking real and imaginary part of previous equation one obtains

$$\kappa = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s-\lambda')\Phi'_{00}\, d\lambda', \tag{18}$$

$$\gamma_1 = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s-\lambda')\Psi'_{0R}\, d\lambda', \tag{19}$$

$$\gamma_2 = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s-\lambda')\Psi'_{0I}\, d\lambda'. \tag{20}$$

- These expressions for the weak field lens quantities are explicitly gauge invariant, since they are given in terms of the curvature components,

Note that the last two equations can be written as

$$\gamma_1 + i\gamma_2 = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s-\lambda')\Psi'_0\, d\lambda'. \tag{21}$$

The general case

The expressions for the lens scalars are reduced to

$$\kappa = \frac{d_l d_{ls}}{d_s} \hat{\Phi}_{00}, \tag{22}$$

$$\gamma_1 + i\gamma_2 = \frac{d_l d_{ls}}{d_s} \hat{\Psi}_0, \tag{23}$$

where

$$\begin{aligned} \hat{\Phi}_{00} &= \int_0^{d_s} \Phi_{00} d\lambda, \\ \hat{\Psi}_0 &= \int_0^{d_s} \Psi_0 d\lambda, \end{aligned} \tag{24}$$

are the projected curvature scalars along the line of sight.

The thin lens approximation II

The axially symmetric case

Let us define

$$\hat{\Psi}_0(J) \quad = \quad -e^{2i\vartheta}\hat{\psi}_0(J), \tag{25}$$

and

$$\gamma_1 + i\gamma_2 = -\gamma e^{2i\vartheta}; \tag{26}$$

then one has

$$\kappa \quad = \quad \frac{d_{ls}d_l}{d_s}\hat{\Phi}_{00}(J), \tag{27}$$

$$\gamma = \frac{d_{ls}d_l}{d_s}\hat{\psi}_0(J), \tag{28}$$

and

$$\boxed{\alpha(J) = J(\hat{\Phi}_{00}(J) + \hat{\psi}_0(J))}. \tag{29}$$

This constitutes a very simple equation for the bending angle expressed in terms of the gauge invariant curvature components in compact form.

Spherically symmetric lenses I

Expressions for the bending angle in terms of energy-momentum components and $M(r)$

$$\boxed{\alpha(J) = J \int_{-d_l}^{d_{ls}} \left[\frac{3J^2}{r^2} \left(\frac{M(r)}{r^3} - \frac{4\pi}{3} \varrho(r) \right) + 4\pi \left(\varrho(r) + P_r(r) \right) \right] dy} ; \tag{30}$$

where $r = \sqrt{J^2 + y^2}$.

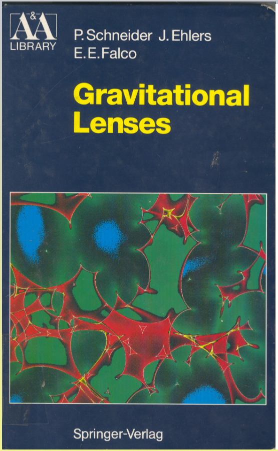
Expressions for the lens scalars in terms of energy-momentum components and $M(r)$

$$\boxed{\kappa = \frac{4\pi d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \left[\rho + P_r + \frac{J^2}{r^2} (P_t - P_r) \right] dy} . \tag{31}$$

$$\boxed{\gamma = \frac{d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \frac{J^2}{r^2} \left[\frac{3M}{r^3} - 4\pi(\rho + P_t - P_r) \right] dy} . \tag{32}$$

Spherically symmetric lenses II

In contrast in text books one finds



The deflection angle. As will be shown in Sect. 4.3, for geometrically-thin lenses the deflection angles of several point masses simply add. Then, one can decompose a general matter distribution into small parcels of mass m_i and write the deflection angle for such a lens as

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sum_i \frac{4G\,m_i}{c^2} \frac{\boldsymbol{\xi} - \boldsymbol{\xi}_i}{|\boldsymbol{\xi} - \boldsymbol{\xi}_i|^2} \quad , \tag{2.16}$$

where $\boldsymbol{\xi}$ describes the position of the light ray in the lens plane, and $\boldsymbol{\xi}_i$ that of the mass m_i.

We can take the continuum limit in (2.16), and replace the sum by an integral. This is most conveniently done by defining $dm = \Sigma(\boldsymbol{\xi}) d^2\xi$, where $d^2\xi$ is the surface element of the lens plane, and $\Sigma(\boldsymbol{\xi})$ is the surface mass density at position $\boldsymbol{\xi}$ which results if the volume mass distribution of the deflector is projected onto the lens plane. We then find

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2} \int_{\mathbb{R}^2} d^2\xi'\, \Sigma(\boldsymbol{\xi}') \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} \quad , \tag{2.17}$$

and the integral extends over the whole lens plane. Let us again point out the conditions which must be fulfilled for (2.17) to be valid: the gravitational fields under consideration must be weak, hence the deflection angle

Figure: Usual equations for gravitational lensing appearing in text books.

Content

Preliminaries

A stationary spherically symmetric spacetime can be expressed in terms of the standard line element

$$ds^2 = a(r)dt^2 - b(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2); \tag{33}$$

where it is convenient to define $\Phi(r)$ and $m(r)$ from

$$a(r) = e^{2\Phi(r)}, \tag{34}$$

and

$$b(r) = \frac{1}{1 - \frac{2m(r)}{r}}. \tag{35}$$

The energy momentum tensor can be given by

$$T_{tt} = \varrho e^{2\Phi(r)}; \tag{36}$$

$$T_{rr} = \frac{P_r}{\left(1 - \frac{2m(r)}{r}\right)}; \tag{37}$$

$$T_{\theta\theta} = P_t\, r^2; \tag{38}$$

An interesting solution of Einstein equations II

$$T_{\varphi\varphi} = P_t\, r^2 \sin(\theta)^2; \tag{39}$$

where we have introduced the notion of radial component P_r and tangential component P_t, due to our general anisotropic asumption.
The (t,t) component of the field equations implies

$$\frac{dm}{dr} = 4\pi r^2 \varrho. \tag{40}$$

Determining the solution

We set the two degrees of freedom from the conditions

$$m(r) = 0, \tag{41}$$

$$P_t(r) = 0; \tag{42}$$

which obviously imply

$$\varrho(r) = 0. \tag{43}$$

This completely determines the geometry of the spacetime.

*

Only P_r is different from zero.

An interesting solution of Einstein equations III

- The metric depends on a minimum radius μ and an integration constant C.
- The curvature shows a logarithmic singular behavior for $r \to \mu$.
- For $r \to \infty$ the curvature tends to zero, but the spacetime is not asymptotically flat[Mor87].
- The energy-momentum tensor satisfies the *strong energy condition*[Wal84] but not the *dominant energy condition*[Wal84]
- The mass is zero; for any of the well behaved notions of quasi-local mass[Pen82].
- The solution can be matched continuously to Minkowski spacetime for some large radius r_0.
- The solution can easily be generalized to contain a constant monopole mass contribution m_0.
- The solution can easily be generalized to contain a monopole mass contribution $m(r)$.

An interesting solution of Einstein equations IV

Rotation curves

Although the mass is zero the geodesic motion is non trivial.

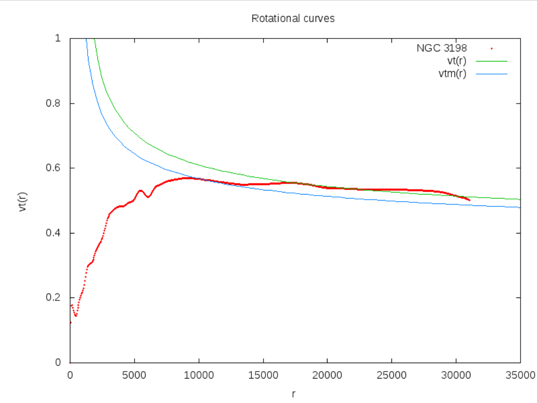


Figure: Rotation curves for NGC 3198 (red), a massless energy-momentum (green) and with a different μ and a small monopole contribution (blue).

An interesting solution of Einstein equations V

What would be the estimate of the Newtonian mass function for this zero mass solution?

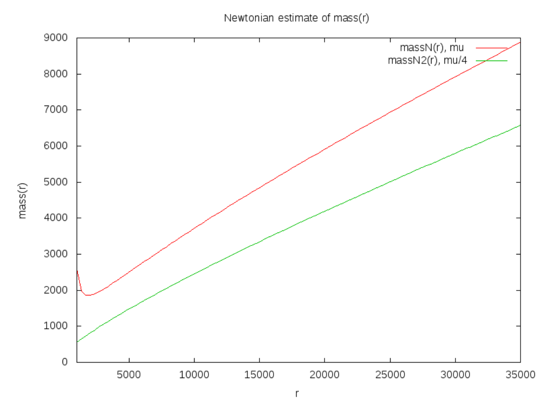


Figure: Newtonian calculation of the mass function $m_N(r)$, for the original μ and for $\mu/4$.

An interesting solution of Einstein equations VI

Gravitational lensing

- Since the spacetime is not asymptotically flat it is necessary to match the solution with an asymptotic metric.
- Preliminary results indicate that the possibility of using the thin lens approximation might depend on the impact parameter of the beam considered.
- Even in the thin lens approximation the integrations must be calculated with numerical techniques.
- It might be necessary, for astrophysical applications, to also match the solution in the interior region.

Content

Final comments I

- We have presented explicit expressions for the bending angle and optical scalars in terms of all the components of the energy-momentum tensor for a variety of cases.
- The incidence of the spacelike components of the energy-momentum tensor is not trivial. However:
  - △ Many articles on gravitational lensing, like references Kubo[JAJ+07] and Gavazzi[G+09] mentioned above, use the simplified equations from text books.
  - △ The NFW profile is calculated from a Newtonian model for dark matter; which implies from the beginning negligible spacelike components of the energy-momentum tensor.
- The zero mass solution presented shows interesting properties found in astrophysical systems where dark matter is needed.

⊛

It might be that the description of dark matter needs for the consideration of the spacelike components of the energy-momentum tensor.

Final comments II

Figure: Are we looking for the keys at the right place?

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