Modeling compact objects by particles in general relativity

Osvaldo M. Moreschi
collaborators: Emanuel Gallo and María Luján Rojas Kaufmann (Lupe)

FaMAF, Universidad Nacional de Córdoba,
Instituto de Física Enrique Gaviola (IFEG), CONICET,
Ciudad Universitaria, (5000) Córdoba, Argentina

FoF, 10th of April, 2013

Content

1. Introduction
   - Particles as isolated objects
   - Particles as point like objects in special relativity
   - The case of a single spherically symmetric object in general relativity
   - On the nature of other previous approaches
   - Motivation for so much work on the motion of 'particles' in general relativity
2. Previous approaches to the notion of particles by other authors
   - Post-Newtonian approaches
   - Methods of self-force
3. Our previous approach based on the harmonic gauge
   - The decomposition of the metric
   - Auxiliary functions or gauge vector
   - The field equations in relaxed covariant form
   - Iterative approximation method
   - The gravitational field from one particle in the first iteration
4. Particle model in the null gauge
   - Linearized gravity
   - General asymptotically flat spacetimes
   - Monopole particles and Robinson-Trautman geometries
   - Properties of the particle model in the null gauge
   - Numeric implementation
5. Final comments

Content

1. Introduction
   - Particles as isolated objects
   - Particles as point like objects in special relativity
   - The case of a single spherically symmetric object in general relativity
   - On the nature of other previous approaches
   - Motivation for so much work on the motion of 'particles' in general relativity
2. Previous approaches to the notion of particles by other authors
   - Post-Newtonian approaches
   - Methods of self-force
3. Our previous approach based on the harmonic gauge
   - The decomposition of the metric
   - Auxiliary functions or gauge vector
   - The field equations in relaxed covariant form
   - Iterative approximation method
   - The gravitational field from one particle in the first iteration
4. Particle model in the null gauge
   - Linearized gravity
   - General asymptotically flat spacetimes
   - Monopole particles and Robinson-Trautman geometries
   - Properties of the particle model in the null gauge
   - Numeric implementation
5. Final comments

Introduction: I

There is more than one way in which one can approach the notion of a particle in general relativity. We mention here two different approaches.

- From the notion of isolated systems; through the study of asymptotically flat spacetimes. Suppose one has the expressions for the general isolated systems in terms of a null frame and some luminosity distance; then neglecting any appearance of quadratic terms in the curvature, one will end up with the most general linearized solution of a compact object with structure. Then one can apply some notion of no-structure, to define a particle.
- Through the study of local solutions, applied to distributions with support on a timelike world line. The corresponding energy-momentum tensor should only take into account mass and spin aspect of the particle.

Particles as isolated objects

When considering a compact object as isolated, in the framework of isolated spacetimes, one realizes that in general one can ascribe a flat background to the spacetime.
More concretely, in the asymptotic region one can always write the metric as

$$g_{ab} = \eta_{ab} + h_{ab}; \tag{1}$$

where η_{ab} is a flat metric and h_{ab} the tensor where all the physical information is encoded. But:

> There are as many flat metrics as there are proper BMS[Sac62, Mor86] supertranslation generators.

- The difficulties in finding appropriate rest frames comes from the existence of gravitational radiation[Mor88, MD98, DM00].
- In the past we have found a way to select rest frames based on the notion of center of mass and intrinsic angular momentum[Mor04].

- For each point at future null infinity, we have[Mor04] a way to single out a unique decomposition of the metric in the form (1), with an appropriately selected flat background.

It is for this reason that gravitational radiation should be the first quantity to be taken into account for the discussion of back reaction on the motion of compact objects.

- Therefore in calculating the appropriate equations of motion for particles we take this as our starting point; so that the root of the difficulties is taken care at the beginning of our approach.
- The structure of the particle is then deduced from the asymptotic structure of isolated systems; namely, asymptotically flat spacetimes.
- This invites to treat the spacetime in terms of a null tetrad adapted to the discussion. We will make use of this later.

Particles as point like objects in special relativity

The notion of a point like object in a fixed background is normally presented through an energy momentum tensor which has support only on a timelike world line.

The first case one should understand if of course the flat background case.

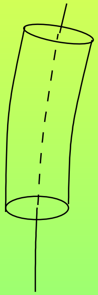


Figure: Small world tube around the world line.

- Consider first the case in which one is considering a distribution of matter that has support within a world tube of finite size. Later we will take the limit in which the tube collapses to a timelike line.
- Having a flat background we have at our disposal the translational Killing vectors $K_{\underline{a}} = \frac{\partial}{\partial x^{\underline{a}}}$, with $\underline{a} = 0, 1, 2, 3$,
- and also the rotational Killing vectors $K_{\underline{ab}}(\xi) = \eta_{\underline{ac}}(x^{\underline{c}} - \xi^{\underline{c}})\frac{\partial}{\partial x^{\underline{b}}} - \eta_{\underline{bc}}(x^{\underline{c}} - \xi^{\underline{c}})\frac{\partial}{\partial x^{\underline{a}}}$.
- The matter is represented by the energy-momentum tensor T_{ab}. Note that we are using the notation a, b, c for abstract indices while underlined indices are numeric. Then, at zero order, the energy-momentum tensor satisfies:

$$\nabla \cdot T = 0 \,. \tag{2}$$

Let K^a denote any of the Killing symmetries, then, as explained elsewhere[GM12], for each symmetry one has a conserved quantity. Namely, let us define the three form

$$D_{abc} = T^d{}_e \, K^e \epsilon_{abcd}; \tag{3}$$

then its exterior derivative is $dD_{abcd} = k\nabla_f \left(T^f{}_e \, K^e\right) \epsilon_{abcd}$, where k is a constant. This exterior derivative vanishes due to the fact that the divergence of T is zero and K is a Killing symmetry. Then if Σ is a three dimensional region, such that the world tube goes through its interior, the quantity

$$Q = \int_\Sigma D; \tag{4}$$

is conserved.

For this reason one defines $P_{\underline{a}}$ by

$$P_{\underline{a}} = \int_\Sigma D_{\underline{a}}; \tag{5}$$

as the components of the conserved total momentum, and also

$$J_{\underline{ab}}(\xi) = \int_\Sigma D_{\underline{ab}}(\xi); \tag{6}$$

as the components of the conserved total angular momentum.

Let us observe that

$$J_{\underline{ab}}(\xi_2) - J_{\underline{ab}}(\xi_1) = \eta_{\underline{ac}}(\xi_1^{\underline{c}} - \xi_2^{\underline{c}})\, P_{\underline{b}} - \eta_{\underline{bc}}(\xi_1^{\underline{c}} - \xi_2^{\underline{c}})\, P_{\underline{a}}; \tag{7}$$

or in other words

$$J^{\underline{ab}}(\xi_2) = J^{\underline{ab}}(\xi_1) + (\xi_1^{\underline{a}} - \xi_2^{\underline{a}})\, P^{\underline{b}} - (\xi_1^{\underline{b}} - \xi_2^{\underline{b}})\, P^{\underline{a}}. \tag{8}$$

Let us emphasize again that $J^{\underline{ab}}(\xi)$ is conserved for any ξ.
But one can consider, for example $\xi_1(\tau_1)$ as a timelike world line with proper time τ_1.
Therefore, calling $\mu_1{}^a = \nabla_{\xi_1}\xi_1^a$ one has

$$\nabla_{\xi_1} J^{\underline{ab}}(\xi_1) = \mu_1{}^{\underline{b}}\, P^{\underline{a}} - \mu_1{}^{\underline{a}}\, P^{\underline{b}}. \tag{9}$$

Note that if one chooses μ_1 to be parallel to P, then

$$\nabla_{\xi_1} J^{\underline{ab}}(\xi_1) = 0. \tag{10}$$

Also, let us note that fixing ξ_2, one has

$$J^{\underline{ab}}(\xi_1)P_{\underline{b}} = J^{\underline{ab}}(\xi_2)P_{\underline{b}} - (\xi_1^{\underline{a}} - \xi_2^{\underline{a}})\, P^2 + (\xi_1^{\underline{b}} - \xi_2^{\underline{b}})P_{\underline{b}}\, P^{\underline{a}}; \tag{11}$$

so that one can always pick a $\xi_1^{\underline{a}}$ so that

$$\boxed{J^{\underline{ab}}(\xi_1) P_{\underline{b}} = 0}. \tag{12}$$

Using the frame in which P only has its timelike component, with all its spacelike components equal to zero; i.e., $P^i = 0$ with $i = 1, 2, 3$; one has

$$\int_\Sigma (x^{\underline{i}} - \xi_1^{\underline{i}}) P^0 \epsilon_\Sigma = \int_\Sigma (x^{\underline{0}} - \xi_1^{\underline{0}}) P^i \epsilon_\Sigma = 0; \tag{13}$$

so that

$$\xi_{\mathrm{cm}}^{\underline{i}} = \frac{1}{M} \int_\Sigma x^{\underline{i}} T^{00} \epsilon_\Sigma; \tag{14}$$

where now Σ is an adapted hypersurface $x^0 =$constant, ϵ_Σ is the volume element, and we have used $P^0 = M$. For this reason we call such a point a center of mass. Then all other points that are translated parallel to P are also center of mass points; which means that the center of mass world line has velocity $\mu = \frac{P}{M}$.

Now, let us consider the case in which the world tube gets smaller until it collapses to the timelike world line z. In this case $x^{\underline{i}}$ is compelled to be evaluated at the world line; and since $P^0 = M$, one has that the center of mass $\xi^{\underline{i}}_{\rm cm}$ is at the world line z. Also this analysis indicates that the velocity of the center of mass u^a is parallel to the momentum P^a; so that, one has

$$P^a = Mu^a \ . \tag{15}$$

The case of a single spherically symmetric object in general relativity

The curious thing in general relativity is that if one starts from an object and compress it in order to obtain a point like object, one ends up with the formation of a black hole; whose size is different from zero.

The most simple black hole is the one represented by Schwarzschild spacetime with metric g_{ab}. This is one of the spacetimes that can be decomposed in the so called Kerr-Schild form; which means that the metric g_{ab} can be expressed in terms of a unique flat metric η_{ab} and a generator of future directed null geodesics ℓ^a, such that

$$g_{ab} = \eta_{ab} - \frac{2Gm}{c^2 r} \ell_a \ell_b. \tag{16}$$

Therefore, given the interior manifold $\mathcal{M}$ one has at ones disposal two metrics. The spacetimes diagrams with respect to both metrics are depicted in figure 2.

Introduction: The case of a single spherically symmetric object in general relativity II

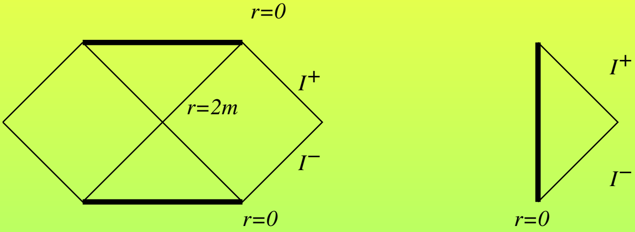
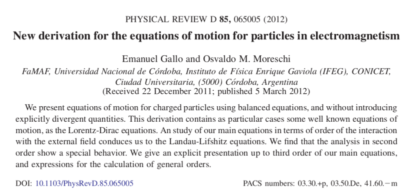


Figure: On the left Schwarzschild metric from the point of view of the Kruskal analytical extension, and on the right the spacetime diagram from the point of view of the flat reference metric. Singularities are drawn with thick lines.

Therefore, although from the point of view of the Schwarzschild metric, the black hole has a size, from the point of view of the reference metric it is a point like object. In astrophysical situations the observer is situated in the asymptotic region of such metrics, so that it is legitimate for him to talk about a point like object.

On the nature of other previous approaches

- Probably one of the first works that tackled de problem of motion of compact objects was that of Einstein-Infeld-Hoffmann[EIH38](See critics in [HG62]). They assumed small velocities and weak fields.
- For a resent review on the problem of motion one can read [PPV11]. They comment on the work of Gralla-Harte-Wald[GHW09], and said that they have made a rigorous derivation of the equations of motion of charged particles.
- It is interesting to note here that later we have been able to derive the most general equation of motion for charged particles in [GM12].


PHYSICAL REVIEW D **85,** 065005 (2012)

New derivation for the equations of motion for particles in electromagnetism

Emanuel Gallo and Osvaldo M. Moreschi

FaMAF, Universidad Nacional de Córdoba, Instituto de Física Enrique Gaviola (IFEG), CONICET, Ciudad Universitaria, (5000) Córdoba, Argentina

(Received 22 December 2011; published 5 March 2012)

We present equations of motion for charged particles using balanced equations, and without introducing explicitly divergent quantities. This derivation contains as particular cases some well known equations of motion, as the Lorentz-Dirac equations. An study of our main equations in terms of order of the interaction with the external field conduces us to the Landau-Lifshitz equations. We find that the analysis in second order show a special behavior. We give an explicit presentation up to third order of our main equations, and expressions for the calculation of general orders.

DOI: 10.1103/PhysRevD.85.065005 PACS numbers: 03.30.+p, 03.50.De, 41.60.−m

Figure: Our article on the motion of charged particles.

- While the first authors have use local information on the particles we have use in our work global information, that takes into account the radiation at infinity.
- The situation is that our equations contain as particular cases the equations of other authors.

The main idea in our approach is to consider the balance of variation of momentum due to the emission of radiation.
Let S be a sphere at future null infinity defined as the asymptotic sphere of the future null cone of a point $Q(\tau)$; and let Σ be the future null cone of this point.

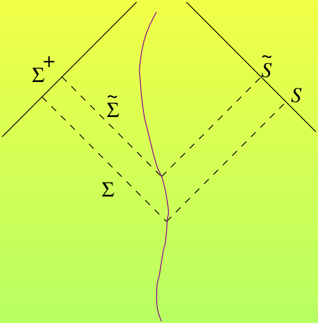


Figure: Two hypersurfaces reaching future null infinity.

- In developing their ideas in the realm of general relativity, they use their approach to the charged particles case, and we use ours. Therefore, we expect to obtain different resulting equations of motion.
- The application of the Gralla-Harte-Wald approach of charged particles to general relativity gives the equations that are known as the MiSaTaQuWa equations.
- In relation to this it is interesting to note that in the review article [PPV11] one can read: "The MiSaTaQuWa equations of motion are not gauge invariant and they cannot by themselves produce a meaningful answer to a well-posed physical question; to obtain such answers it is necessary to combine the equations of motion with the metric perturbation $h_{\alpha\beta}$ so as to form gauge-invariant quantities that will correspond to direct observables."

Motivation for so much work on the motion of ‘particles’ in general relativity

Today there are several interferometric gravitational wave observatories constructed in the world and they are supposed to detect gravitational radiation in the near future. The study of the observed data[A+09, A+10, A+11] use:

- Time coincidence to identify possible gravitational waves.
- Template banks generated from post-Newtonian (PN) calculations extended to the Schwarzschild inner most stable circular orbits.
- Calculations from numerical relativity (NR) of waveforms in the late inspiral and merger of binary black holes systems.
- However: “it is infeasible to use the NR simulations directly as search templates”[A+11].
- Therefore they[A+11] use “phenomenological waveforms”.

Introduction: Motivation for so much work on the motion of 'particles' in general relativity II

- The previous situation motivates the study for extending the particle paradigm to the relativistic regime, in which the gravitational radiation effects are taken into account.

Several related questions arise:

- what is the convenient structure that one should assume for the corresponding particle?
- how should one determine the corresponding equation of motion?

We here present a different approach to the particle paradigm, where no restriction on the weakness of the sources are imposed, nor slow motion is assumed. In particular we present a closed model for the binary system in general relativity.

Content

1. Introduction
   - Particles as isolated objects
   - Particles as point like objects in special relativity
   - The case of a single spherically symmetric object in general relativity
   - On the nature of other previous approaches
   - Motivation for so much work on the motion of 'particles' in general relativity
2. Previous approaches to the notion of particles by other authors
   - Post-Newtonian approaches
   - Methods of self-force
3. Our previous approach based on the harmonic gauge
   - The decomposition of the metric
   - Auxiliary functions or gauge vector
   - The field equations in relaxed covariant form
   - Iterative approximation method
   - The gravitational field from one particle in the first iteration
4. Particle model in the null gauge
   - Linearized gravity
   - General asymptotically flat spacetimes
   - Monopole particles and Robinson-Trautman geometries
   - Properties of the particle model in the null gauge
   - Numeric implementation
5. Final comments

Post-Newtonian approaches

It is very natural to consider as a first approximation to the problem of particles, or compact objects, the situation in which fields are weak and velocities involved are small. These are the main assumptions of the Post-Newtonian approximations. All quantities are considered as order of powers $O((\frac{v}{c})^q)$. In general one talks about an n PN order, when the power involved is $q = 2n$.

In addition to expand the field equations using these approximations, one can also expand the equations of motion, that arise as integrability conditions of the field equations. In this case one also makes use of an n PN expansion. However, it has been pointed out[Sch09] that in general one can not expand the solutions of the field equations using the later.

In these approaches, the effects of gravitational radiation appear at the $O(\frac{1}{c^5})$ order. It is for this reason that one can have a Lagrangian description of a particle system of order $O(\frac{v^4}{c^4})$; since at this order, one would have conservation of energy, total momentum and total angular momentum.

We reproduce here the equations of motion of reference [BFP98]

$$
\begin{aligned}
\frac{dv_1^i}{dt} = &- \frac{Gm_2}{r_{12}^2} n_{12}^i + \frac{Gm_2}{r_{12}^2 c^2} \Bigg\{ v_{12}^i \left[4(n_{12} v_1) - 3(n_{12} v_2)\right] \\
&+ n_{12}^i \left[-v_1^2 - 2v_2^2 + 4(v_1 v_2) + \frac{3}{2}(n_{12} v_2)^2 + 5\frac{Gm_1}{r_{12}} + 4\frac{Gm_2}{r_{12}} \right] \Bigg\} \\
&+ \frac{Gm_2}{r_{12}^2 c^4} n_{12}^i \Bigg\{ \left[-2v_2^4 + 4v_2^2 (v_1 v_2) - 2(v_1 v_2)^2 + \frac{3}{2} v_1^2 (n_{12} v_2)^2 + \frac{9}{2} v_2^2 (n_{12} v_2)^2 \right. \\
&\left. -6(v_1 v_2)(n_{12} v_2)^2 - \frac{15}{8}(n_{12} v_2)^4 \right] + \frac{Gm_1}{r_{12}} \left[-\frac{15}{4} v_1^2 + \frac{5}{4} v_2^2 - \frac{5}{2}(v_1 v_2) \right. \\
&\left. + \frac{39}{2}(n_{12} v_1)^2 - 39(n_{12} v_1)(n_{12} v_2) + \frac{17}{2}(n_{12} v_2)^2 \right] \\
&+ \frac{Gm_2}{r_{12}} \left[4v_2^2 - 8(v_1 v_2) + 2(n_{12} v_1)^2 - 4(n_{12} v_1)(n_{12} v_2) - 6(n_{12} v_2)^2 \right] \\
&+ \frac{G^2}{r_{12}^2} \left[-\frac{57}{4} m_1^2 - 9m_2^2 - \frac{69}{2} m_1 m_2 \right] \Bigg\} \\
&+ \frac{Gm_2}{r_{12}^2 c^4} v_{12}^i \Bigg\{ v_1^2 (n_{12} v_2) + 4v_2^2 (n_{12} v_1) - 5v_2^2 (n_{12} v_2) - 4(v_1 v_2)(n_{12} v_1) \\
&+ 4(v_1 v_2)(n_{12} v_2) - 6(n_{12} v_1)(n_{12} v_2)^2 + \frac{9}{2}(n_{12} v_2)^3 \\
&+ \frac{Gm_1}{r_{12}} \left[-\frac{63}{4}(n_{12} v_1) + \frac{55}{4}(n_{12} v_2) \right] + \frac{Gm_2}{r_{12}} \left[-2(n_{12} v_1) - 2(n_{12} v_2) \right] \Bigg\}
\end{aligned}
\tag{17}
$$

Very often the equations appear in a complicated form, due to the fact that accelerations appear on the right hand side. This forces to make some cleaning until one has equations that can be used as in classical mechanics.
One can see that the dynamics assumes an absolute Newtonian time; so that the retardation relativistic effects are not included in these descriptions.
The methods of Post-Newtonian approximations do not take into account the back reaction to the equation of motion due to the gravitational radiation.
We will not discuss the so called method of effective field theory[GR06] since they are a technique to calculate post-Newtonian terms.

Methods of self-force

There is another approach which is associated to the notion of self-force. In this approaches the idea is to consider first effects due to the perturbation that a particle would produce in the background spacetime. For this reason it is assumed from the beginning that the masses of the particles are small, in an appropriate sense. The metric is assumed to be expressible in terms of

$$g_{ab} = \mathbf{g}_{ab} + h_{ab}; \tag{18}$$

where h_{ab} is a perturbation to the background metric $\mathbf{g}_{ab}$; which is assumed to be a solution of the field equations.

- The dynamics of the particle can be deduced[Poi 6] from the geodesic equation of the total metric.
- This means that in this approach, one also neglects the effects due to the gravitational radiation involving the motion of the particles.

The perturbation h_{ab} is a solution of the linearized problem. In first instance the equation of motion is given by

$$\frac{du^a}{d\tau} = \frac{1}{2}(-\mathbf{g}_{ab} + u^a u^b)(2h_{bc;d} - h_{cd;b})u^c u^d; \tag{19}$$

where τ is the proper time with respect to $\mathbf{g}_{ab}$, and the covariant derivatives are also with respect to the background metric.
This equation is singular, since one has to evaluate h_{ab} at the particle's world line. The method then recurs to a decomposition in terms of a singular and regular terms; so that finally one arrives at the equations of motion

$$\frac{du^a}{d\tau} = \frac{1}{2}(-\mathbf{g}_{ab} + u^a u^b)(2h^{\text{tail}}_{bcd} - h^{\text{tail}}_{cdb})u^c u^d; \tag{20}$$

where

$$h^{\text{tail}}_{bcd} = 4m \int_{\infty}^{\tau^-} \nabla_d \left(G_{+bcb'c'} - \frac{1}{2}\mathbf{g}_{bc} G_{+e\ \ b'c'}^{\ \ e} \right) u^{b'} u^{c'}; \tag{21}$$

and $G_{+bcb'c'}$ is the retarded Green function of the linear problem over the metric $\mathbf{g}_{ab}$.

Content

1 Introduction
- Particles as isolated objects
- Particles as point like objects in special relativity
- The case of a single spherically symmetric object in general relativity
- On the nature of other previous approaches
- Motivation for so much work on the motion of ‘particles’ in general relativity

2 Previous approaches to the notion of particles by other authors
- Post-Newtonian approaches
- Methods of self-force

3 Our previous approach based on the harmonic gauge
- The decomposition of the metric
- Auxiliary functions or gauge vector
- The field equations in relaxed covariant form
- Iterative approximation method
- The gravitational field from one particle in the first iteration

4 Particle model in the null gauge
- Linearized gravity
- General asymptotically flat spacetimes
- Monopole particles and Robinson-Trautman geometries
- Properties of the particle model in the null gauge
- Numeric implementation

5 Final comments

The standard approach is to treat the field equations through the approximation based on the relaxed covariant form of the field equations, as studied in [GM2b]. We start by presenting a summary of its elements.

The decomposition of the metric

The metric is decomposed as the sum of a flat background plus the physical term, where all the information is encoded.

$$g_{ab} = \eta_{ab} + h_{ab}. \tag{22}$$

Then the torsion free metric connection ∇_a of the metric g_{ab} can be expressed it terms of the torsion free metric connection ∂_a of the metric η_{ab}; so that of v is an arbitrary vector one has

$$\nabla_a v^b = \partial_a v^b + \Gamma_a{}^b{}_c v^c; \tag{23}$$

and one can prove that

$$\Gamma_a{}^c{}_b = \frac{1}{2} g^{cd} \left(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}\right) = \Gamma_b{}^c{}_a. \tag{24}$$

The relation between Γ and the curvature tensor can be calculated from

$$\begin{aligned}[\nabla_a, \nabla_b] v^d &= \Theta_{abc}{}^d v^c + \left(\partial_a \Gamma_b{}^d{}_c - \partial_b \Gamma_a{}^d{}_c + \Gamma_a{}^d{}_e \Gamma_b{}^e{}_c - \Gamma_b{}^d{}_e \Gamma_a{}^e{}_c\right) v^c \\ &= R_{abc}{}^d v^c;\end{aligned} \tag{25}$$

Our previous approach based on the harmonic gauge II

where Θ is the curvature of the ∂_a connection. Then the Ricci tensor can be calculated from

$$\begin{aligned} R_{ac} &\equiv R_{abc}{}^{b} \\ &= \Theta_{ac} + \partial_a \Gamma^{\ b}_{b\ c} - \partial_b \Gamma^{\ b}_{a\ c} + \Gamma^{\ b}_{a\ e} \Gamma^{\ e}_{b\ c} - \Gamma^{\ b}_{b\ e} \Gamma^{\ e}_{a\ c}; \end{aligned} \tag{26}$$

where Θ_{ac} is the Ricci tensor of the connexion ∂_a.

Auxiliary functions or gauge vector

Let us consider four independent auxiliary functions x^μ, with $\mu = 0, 1, 2, 3$. Then let us observe that

$$g^{ab} \nabla_a \nabla_b x^\mu = g^{ab} \nabla_a \partial_b x^\mu = g^{ab} \partial_a \partial_b x^\mu - g^{ab} \Gamma^{\ c}_{a\ b} \partial_c x^\mu. \tag{27}$$

Then, if $l_\mu^{\ e}$ denotes the inverse of $\partial_c x^\mu$, which exists by assumption of the independence of the set x^μ, one has

$$g^{ab} \Gamma^{\ c}_{a\ b} = -\left(g^{ab} \nabla_a \nabla_b x^\mu - g^{ab} \partial_a \partial_b x^\mu \right) l_\mu^{\ c} = H^\mu l_\mu^{\ c}; \tag{28}$$

where we are using

$$H^\mu = -g^{ab} \nabla_a \nabla_b x^\mu + g^{ab} \partial_a \partial_b x^\mu. \tag{29}$$

Alternatively, let us define the gauge vector

$$\mathscr{H}^c = H^\mu l_\mu^{\ c}; \tag{30}$$

which implies

$$H^\mu = \partial_c x^\mu \ \mathscr{H}^c = \mathscr{H}(x^\mu); \tag{31}$$

then one has

$$g^{ab}\Gamma_{a\ b}^{\ c} = \mathscr{H}^c. \tag{32}$$

Then the Ricci tensor can be expressed by

$$\boxed{\begin{aligned} R_{ac} = \Theta_{ac} &+ \frac{1}{2}g^{bd}\left(\Theta_{bad}{}^{e}h_{ec} + \Theta_{bcd}{}^{e}h_{ea} + 2\Theta_{bca}{}^{e}h_{ed}\right) \\ &+ \frac{1}{2}g^{bd}\partial_b\partial_d h_{ac} - \partial_{(a}\left(g_{c)e}\mathscr{H}^e\right) + g_{ed}\Gamma_{a\ c}^{\ e}\mathscr{H}^d \\ &- g^{bf}g_{ed}\Gamma_{a\ f}^{\ d}\Gamma_{b\ c}^{\ e} - \frac{1}{2}\left(\Gamma_a^{\ bd}\Gamma_{bcd} + \Gamma_c^{\ bd}\Gamma_{bad}\right). \end{aligned}} \tag{33}$$

Given the vector field $\mathscr{H}^c$, we have seen that H^μ can be interpreted as $\mathscr{H}(x^\mu)$. Therefore, one can think that the four auxiliary functions x^μ are to be calculated from the equation

$$-g^{ab}\nabla_a\nabla_b x^\mu + g^{ab}\partial_a\partial_b x^\mu = \mathscr{H}(x^\mu). \tag{34}$$

Alternatively one could think in given four independent functions x^μ. Then calculate the left hand side of (34), and look for a vector field $\mathscr{H}^c$ so that (34) is satisfied.

The field equations in relaxed covariant form

The field equations are

$$R_{ac} = -8\pi\kappa\left(T_{ac} - \frac{1}{2}g_{ac}g^{bd}T_{bd}\right). \tag{35}$$

In writing equation (33) in a coordinate frame, without any reference to η, one would obtain the analogous expression without the Θ terms, and where all the appearance of ∂ derivatives are replace by coordinate derivatives.

Suppose that one solves (35) for a given vector field H^μ. Also assume that one can solve for the functions x^μ such that $g^{ab}\nabla_a\nabla_b x^\mu = H^\mu$. Then, let build a flat metric η so that $g^{ab}\partial_a\partial_b x^\mu = 0$; which in particular can be satisfied if the x^μ's are thought as

Our previous approach based on the harmonic gauge V

Cartesian coordinates of η. In this way one would obtain $H^{\mu} I^{e}_{\mu} = g^{ab}\Gamma_{a\ b}^{\ e}$, and so have a solution of the field equations.
It also might be of interest to researchers in numerical relativity, since it provides the possibility to use any coordinate system; i.e., not necessarily an harmonic one.
Alternatively one can use the form of the field equations in terms of a slight different logic.
If we use the expression of the Ricci tensor as given by (33) in (35), namely

$$\boxed{\begin{aligned} &\frac{1}{2} g^{bd}\partial_b\partial_d h_{ac} - \partial_{(a}\left(g_{c)e}\mathscr{H}^e\right) + g_{ed}\Gamma_{a\ c}^{\ d}\mathscr{H}^e \\ &\qquad + \Theta_{ac} + \frac{1}{2} g^{bd}\left(\Theta_{bad}{}^{e} h_{ec} + \Theta_{bcd}{}^{e} h_{ea} + 2\Theta_{bca}{}^{e} h_{ed}\right) \\ &\qquad - g^{bf} g_{ed}\Gamma_{a\ f}^{\ d}\Gamma_{b\ c}^{\ e} - \frac{1}{2}\left(\Gamma_{a}^{\ bd}\Gamma_{bcd} + \Gamma_{c}^{\ bd}\Gamma_{bad}\right) \\ &\quad = -8\pi\kappa\left(T_{ac} - \frac{1}{2} g_{ac} g^{bd} T_{bd}\right); \end{aligned}} \tag{36}$$

we will refer to this as the relaxed field equations[WW80], where one has not assumed that $\mathscr{H}^c$ is $g^{ab}\Gamma_{a\ b}^{\ c}$.

Our previous approach based on the harmonic gauge VI

Using the standard harmonic gauge technique, one would say: solve the relaxed field equation in the coordinate frame, with $H^{\mu} = 0$, and then require the equation

$$g^{bd} \nabla_b \nabla_d x^{\mu} = 0. \tag{37}$$

In the standard approach one makes use of coordinate basis; therefore the previous statement would be the complete story. At this point it is important to notice that if we have the solutions x^{μ} from (37) then, on constructing η with this as a Cartesian coordinate system, one would obtain $H^{\mu} = 0$.

Several authors have indicated that actually to request equation (37) is equivalent[EIH38, And73, WW80] to demand

$$g^{ab} \nabla_a T_{bc} = 0. \tag{38}$$

When dealing with Einstein equations in the relaxed form, and treating the vacuum case, equation (38) is understood as the condition that the divergence of the Einstein tensor must be zero (which of course is identically zero in the non relaxed form).
In reference [GM2b] we have related this approach to the results of Friedrich[Fri85]

Iterative approximation method

The idea is to express (35) and eventually (37) in the form

$$\varphi\,\eta^{ab}\partial_a\partial_b f = S(f); \tag{39}$$

where $\varphi\,\eta^{ab}$ is the term proportional to η^{ab} that is contained in g^{ab}. This equation can also be expressed by

$$\eta^{ab}\partial_a\partial_b(\varphi\, f) = s(\varphi\, f) + S(f); \tag{40}$$

where

$$s(\varphi\, f) \equiv \eta^{ab}\partial_a\partial_b(\varphi\, f) - \varphi\,\eta^{ab}\partial_a\partial_b f. \tag{41}$$

Now one would like to solve equation (40) by iterations.
Let us define the sets $f^{(j)}$ such that for $j = 0$, one takes $h = 0$, x^μ to be harmonic functions of the metric η and $\varphi = 1$; and for $j > 0$, $f^{(j)}$ is the solution of

$$\eta^{ab}\partial_a\partial_b(\varphi^{(j-1)}\, f^{(j)}) = s(\varphi^{(j-1)}\, f^{(j-1)}) + S(f^{(j-1)}). \tag{42}$$

using the retarded Green function.

The gravitational field from one particle in the first iteration

Let us consider a massive point particle with mass m_A describing, in a flat space-time (M^0, η_{ab}), a curve C which in a Cartesian coordinate system x^a reads

$$x^\mu = z^\mu(\tau), \tag{43}$$

with τ meaning the proper time of the particle along C.

The unit tangent vector to C, with respect to the flat background metric is

$$u^\mu = \frac{dz^\mu}{d\tau}, \tag{44}$$

that is, $\eta(u,u) = 1$. Now, for each point $Q = z(\tau)$ of C, we draw a future null cone $\mathfrak{C}_Q$ with vertex in Q. If we call x^μ_P the Minkowskian coordinates of an arbitrary point on the cone $\mathfrak{C}_Q$, then we can define the retarded radial distance on the null cone by

$$r = u_\mu \left(x^\mu_P - z^\mu(\tau)\right). \tag{45}$$

The energy momentum tensor $T^{(0)}_{ab}$ of a point particle is proportional to $mv_a v_b$; where m is the mass and v^a its four velocity. We are distinguishing between the unite tangent vector u^a and the four velocity vector v^a, because we would like to consider the possibility to normalize the vector v with respect to a different metric. In order to

Our previous approach based on the harmonic gauge IX

represent a point particle $T^{(0)}_{ab}$ must also be proportional to a three dimensional delta function that has support on the world line of the particle.
We will suppose that the particle does not have multipolar structure. Then, given an arbitrary Minkowskian frame (x^0, x^1, x^2, x^3), we will express the energy momentum by

$$\begin{aligned} T^{(0)ab}(x^0 = z^0(\tau_0), x^1, x^2, x^3) =& m_A v^a(\tau_0) v^b(\tau_0) \\ & \frac{\delta(x^1 - z^1(\tau_0))\delta(x^2 - z^2(\tau_0))\delta(x^3 - z^3(\tau_0))}{u^0(\tau_0)}; \end{aligned} \tag{46}$$

however see below, since after considering the equations of motion we will update this form of the energy-momentum tensor to include corrections of order $\mathscr{O}(m_B)$. This will in turn imply an update of the first order solution to the field equations. In this way the first order solution will imply at least second order terms in the masses, and therefore in the gravitational constant G.
The form of the differential equation comes from the study of the distribution in a small world tube surrounding the world line of the particle.
Then the solution to the linear problem is

$$h^{(1)}_{ab} = -4 m_A \frac{v_a v_b - \frac{1}{2}\eta_{ab}}{r}; \tag{47}$$

so that in general

$$g^{(1)}_{ab} = \left(1 + \frac{2m_A}{r}\right)\eta_{ab} - \frac{4m_A}{r} v_a v_b. \tag{48}$$

In these equations we have considered the definition

$$v_a \equiv \eta_{ab} v^b; \tag{49}$$

however it should be emphasized that the vector v^b is not normalized with the flat metric η as we will see below.
The inverse of this metric is

$$g^{(1)ab} = \frac{1}{1 + \frac{2m_A}{r}}\eta^{ab} + \frac{\frac{4m_A}{r}}{1 - \left(\frac{2m_A}{r}\right)^2} v^a v^b. \tag{50}$$

The balanced equations of motion are:

$$\boxed{\frac{dP_{(A)}{}^a}{d\tau_A} = -\frac{1}{G}\mathcal{F}^b_{\tau_A}\left(\delta^a_b - v_{Ab} v^a_A\right)}, \tag{51}$$

and

$$\boxed{\frac{dP_{(B)}{}^{a}}{d\tau_B} = -\frac{1}{G}\mathcal{F}^{b}_{\tau_B}\left(\delta^a_b - v_{Bb}v^a_B\right)}, \tag{52}$$

where $\mathcal{F}^{a}_{\tau_A}$ is the flux of gravitational radiation, δ^a_b is the identity tensor and we have introduced the G factor to emphasize the difference between the local momentum $P_{(A)}$ and the global Bondi momentum $\mathcal{P}_A$. Note that due to the nature of the momentum vector, any correction to the zero order equation of motion must be orthogonal to the four velocity.

How does these equations affect the balance of total Bondi momentum? For a system in which slow motion is a good approximation, one can expect that $G\,P^a_{(A)} \approx \mathcal{P}^a_A$; from which one would conclude that the model satisfies the balance equation at the center of mass at the order $(p = 2, q = 2)$ in the slow motion limit case.

This approach have several encouraging properties; for example:

- It presents no difficulties in an expansion of the field equation in an approximation scheme.
- It allows to deal with objects of high velocities.
- It allows to discuss objects with arbitrary masses.

- It satisfies the balanced equation of motion for each particle.

However, let us note that

$$g^{(1)}_{ab} v^a v^b = \left(1 - \frac{2m_A}{r}\right) \eta_{ab} v^a v^b; \tag{53}$$

is a scalar and vanishes at $r = 2m_A$.
So, we have started with a vector v^a that was assumed to be timelike with respect to the flat background metric η_{ab}; but, the vector field that comes from the first order solution, turns out to be null at some points, with respect to the first order metric $g^{(1)}_{ab}$. This constitutes a complication for the physical interpretation of the ideology that the zero order equations of motion should be corrected by the balance of the gravitational energy radiated. Since one imagines that there should be a null hypersurface emanating from the position of the particle, reaching future null infinity, and therefore providing with a connection between the local behavior and the asymptotic properties of the radiation field. But, then one would find the the vector field v^a, would become null as one gets close to the particle; which is difficult to interpret.

Content

1. Introduction
   - Particles as isolated objects
   - Particles as point like objects in special relativity
   - The case of a single spherically symmetric object in general relativity
   - On the nature of other previous approaches
   - Motivation for so much work on the motion of 'particles' in general relativity
2. Previous approaches to the notion of particles by other authors
   - Post-Newtonian approaches
   - Methods of self-force
3. Our previous approach based on the harmonic gauge
   - The decomposition of the metric
   - Auxiliary functions or gauge vector
   - The field equations in relaxed covariant form
   - Iterative approximation method
   - The gravitational field from one particle in the first iteration
4. Particle model in the null gauge
   - Linearized gravity
   - General asymptotically flat spacetimes
   - Monopole particles and Robinson-Trautman geometries
   - Properties of the particle model in the null gauge
   - Numeric implementation
5. Final comments

Particle model in the null gauge I

The seeds in this method are solutions to the linear problem of particles but in the null gauge; so that we start by viewing how to make the calculation of the spacetime associated to a particle.

Linearized gravity

Let us denote with u a null function that is constant on the future null cones, emanating from the particle at first order.

Using the null polar coordinate system $(x^0, x^1, x^2, x^3) = \big(u, r, (\zeta + \bar{\zeta}), \frac{1}{i}(\zeta - \bar{\zeta})\big)$ one can express the null tetrad as:

$$\ell_a = (du)_a \tag{54}$$

$$\ell^a = \left(\frac{\partial}{\partial r}\right)^a \tag{55}$$

$$m^a = \xi^i \left(\frac{\partial}{\partial x^i}\right)^a \tag{56}$$

$$\overline{m}^a = \bar{\xi}^i \left(\frac{\partial}{\partial x^i}\right)^a \tag{57}$$

$$n^a = \left(\frac{\partial}{\partial u}\right)^a + U \left(\frac{\partial}{\partial r}\right)^a + X^i \left(\frac{\partial}{\partial x^i}\right)^a \tag{58}$$

Particle model in the null gauge II

with $i = 2, 3$.
This is a null tetrad so that

$$g_{ab}\ \ell^a\ n^b = 1, \tag{59}$$

and

$$g_{ab}\ m^a\ \bar{m}^b = -1; \tag{60}$$

and all other scalar products are zero.
Since the metric can be expressed by

$$g_{ab} = \ell_a\ n_b + n_a\ \ell_b - m_a\ \overline{m}_b - \overline{m}_a\ m_b; \tag{61}$$

one has the relations

$$g_{ur} = 1, \tag{62}$$
$$g_{rr} = 0, \tag{63}$$
$$g_{ri} = 0, \tag{64}$$
$$g_{uu} = -2U + X^i X^j g_{ij}, \tag{65}$$
$$g_{ui} = -g_{ij} X^j, \tag{66}$$
$$g_{ij} = (g^{ij})^{-1} = -d\epsilon_{ik}\epsilon_{jl}(\xi^k \bar{\xi}^l + \bar{\xi}^k \xi^l); \tag{67}$$

with $i, j, k, l = 2, 3$, $d = \det(g_{ij})$, $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{23} = 1$. In particular, defining the quantity

$$\lambda = \epsilon_{ij}\xi^i\bar{\xi}^j; \tag{68}$$

one has that

$$d = \frac{1}{|\lambda|^2}. \tag{69}$$

The inverse metric is given by

$$g^{uu} = 0, \tag{70}$$

$$g^{ur} = 1, \tag{71}$$

$$g^{ui} = 0, \tag{72}$$

$$g^{rr} = 2\,U, \tag{73}$$

$$g^{ri} = X^i, \tag{74}$$

$$g^{ij} = -(\xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j). \tag{75}$$

Particle model in the null gauge IV

Let us use the parameter γ to denote the order of the gravitational constant. Then, the fact that $g_{ab}(\gamma = 0)$ is the flat metric implies that the linear null tetrad components have the form

$$U = U_0 + \gamma U_\gamma, \quad U_0 = \frac{\dot{V}_M}{V_M} r - \frac{1}{2}, \tag{76}$$

$$\xi^2 = \frac{\sqrt{2} P_0 V_M (1 + \gamma V_\gamma)}{r}, \tag{77}$$

$$X^j = O(\gamma); \tag{78}$$

where a dot means $\partial/\partial u$, $P_0 = (1 + \zeta\bar{\zeta})/2$, and V_M is given by the following expression

$$V_M = \hat{l}^a V^b \eta_{ab}, \tag{79}$$

where $(\eta_{\mu\nu}) = diag(1, -1, -1 - 1)$, $V^\mu = V^\mu(u)$ is a four timelike vector in Minkowski space-time, which depends only on u and satisfies the normalization

$$V^a V^b \eta_{ab} = 1,$$

and the null vector $\hat{l}^a$ is given by

$$(\hat{l}^\nu) = \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta\bar{\zeta})}, \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}\right). \tag{80}$$

Particle model in the null gauge V

We distinguish between the abstract indices $a, b, \cdots$ and the numeric indices $\mu, \nu, \cdots = 0, 1, 2, 3$.
The relation between these null coordinates and the Cartesian ones y^μ is the following

$$y^\mu = \varphi^\mu(u) + r\ell^\mu(u, x^2, x^3),$$

where $\varphi^\mu(u)$ is a world line with unit tangent vector V^μ

$$V^\mu = \frac{d\varphi^\mu(u)}{du},$$

and the Cartesian components of the vector l are

$$\ell^\mu = \frac{\hat{l}^\mu}{V_M}.$$

The vector field ℓ^a generates the future null geodesics with origin at points of the world line φ^μ; note that we are taking the variable u as the proper time of this world line.

Particle model in the null gauge VI

It is convenient to study the vacuum equations ordered by the components of the Ricci tensor in terms of the vector field ℓ^a, i.e. in the following order: $\Phi_{00} = 0$, $\Phi_{01} = 0$, $\Phi_{11} = 0$, $\Lambda = 0$, $\Phi_{02} = 0$, $\Phi_{12} = 0$ and $\Phi_{22} = 0$ (we make use of the spin coefficient formalism [GHP73]).

General asymptotically flat spacetimes

Let now u denote a null hypersurface that contains future directed null geodesics that reach future null infinity. Then we can use the same null tetrad prescription, that we used before, but now adapted to this null congruence.

The components ξ^i, U and X^i are:

$$\xi^2 = \frac{\xi_0^2}{r} + O\left(\frac{1}{r^2}\right), \qquad \xi^3 = \frac{\xi_0^3}{r} + O\left(\frac{1}{r^2}\right), \tag{81}$$

with

$$\xi_0^2 = \sqrt{2} P_0 \, V, \qquad \xi_0^3 = -i\xi_0^2, \tag{82}$$

where $V = V(u, \zeta, \bar{\zeta})$ and the square of $P_0 = \frac{(1+\zeta\bar{\zeta})}{2}$ is the conformal factor of the unit sphere;

$$U = rU_{00} + U_0 + \frac{U_1}{r} + O\left(\frac{1}{r^2}\right), \tag{83}$$

Particle model in the null gauge VII

where

$$U_{00} = \frac{\dot{V}}{V}, \quad U_0 = -\frac{1}{2} K_V, \quad U_1 = -\frac{\Psi_2^0 + \overline{\Psi}_2^0}{2}, \tag{84}$$

where K_V is the curvature of the 2-metric

$$dS^2 = \frac{1}{V^2\, P_0^2}\; d\zeta\; d\bar{\zeta}; \tag{85}$$

where the regular conformal metric restricted to scri is precisely $\tilde{g}\mid_{\mathcal{I}^+} = -dS^2$. In terms of the edth operator $\eth_V$ of the sphere (85) the curvature K_V is given by

$$K_V = \frac{2}{V}\, \bar{\eth}_V \eth_V\, V - \frac{2}{V^2}\, \eth_V V\, \bar{\eth}_V V + V^2. \tag{86}$$

Finally, the other components of the vector n^a have the asymptotic form

$$X^2 = O\left(\frac{1}{r^2}\right), \qquad X^3 = O\left(\frac{1}{r^2}\right). \tag{87}$$

One can see that the previous expressions can be consider as a subset of the present equations; since the first is expressing an asymptotically flat spacetime.

Monopole particles and Robinson-Trautman geometries

- There is an interesting connection between the particular case of monopole particles and the so called Robinson-Trautman geometries[DMG96]; which are generalizations of Robinson-Trautman spacetimes.

Robinson-Trautman[RT62] spacetimes (RT) have been very useful for estimating the total gravitational radiation in the head-on black hole collision[MD96][Mor99][AHS$^+$93]. In reference [MD96] we have applied these geometries to the description of the total energy radiated in the head-on black hole collision with equal mass; and it was shown that our calculations agree remarkably well with the numerical exact calculations of Anninos et.al.[AHS$^+$93]. The case of unequal mass black hole collision, was treated numerically in reference [AB98]; and our technique based on the use of the RT geometries[Mor99] showed again an impressive agreement with the exact calculations. If one wants to generalize these estimates to the case of the black hole collision with orbital angular momentum it is necessary to consider spacetimes with total angular momentum. The Robinson-Trautman vacuum solutions are algebraically special spacetimes which are characterized by the existence of a congruence of diverging null geodesics without shear and twist. This implies the existence of a prefered family of null hypersurfaces; which provides a set of sections at future null infinity. The angular momentum calculated on

these sections is found to be zero. (In a Bondi frame the asymptotic NP quantities Ψ_1^0 and σ_0 are zero; as a consequence the angular momentum vanishes independently of the definition used. See for example reference [Mor86] and references therein.)
The line element of these metrics can be expressed by:

$$ds^2 = \left(-2Hr + K - 2\frac{M(u)}{r}\right) du^2 + 2\, du\, dr - \frac{r^2}{P^2} d\zeta\, d\bar{\zeta}, \tag{88}$$

where $P = P(u, \zeta, \bar{\zeta})$, $H = \frac{\dot{P}}{P}$, $K = \Delta \ln P$, a doted quantity denotes its time derivative, a bar means complex conjugate and Δ is the two-dimensional Laplacian for the two-surfaces $u = \texttt{constant}$, $r = \texttt{constant}$ with line element

$$dS^2 = \frac{1}{P^2}\, d\zeta\, d\bar{\zeta}; \tag{85}$$

where we are using complex stereographic coordinates $(\zeta, \bar{\zeta})$ for the sphere.
It is usually convenient to describe this line element in terms of the line element of the unit sphere; this is done by expressing P as the product $P = V(u, \zeta, \bar{\zeta}) P_0(\zeta, \bar{\zeta})$, where P_0 is the value of P for the unit sphere.
It is natural in case to use the null tetrad adapted to this geometry. So that ℓ denotes the vector field that generates the null congruence, then $\ell = du$, $\ell(r) = 1$, $\ell(\zeta) = 0$ and

$\ell(\bar{\zeta}) = 0$. It is convenient to use the parameterization u such that the mass parameter $M(u) = M_0 = \texttt{constant}$.
Then, the components ξ^i, U and X^i are:

$$\xi^0 = 0, \quad \xi^2 = \frac{\xi_0^2}{r}, \quad \xi^3 = \frac{\xi_0^3}{r}, \tag{89}$$

with

$$\xi_0^2 = \sqrt{2} P_0 \, V, \qquad \xi_0^3 = -i\xi_0^2; \tag{90}$$

$$U = rU_{00} + U_0 + \frac{U_1}{r}, \tag{91}$$

where

$$U_{00} = \frac{\dot{V}}{V}, \quad U_0 = -\frac{1}{2} K_V, \quad U_1 = -\frac{\Psi_2^0 + \bar{\Psi}_2^0}{2}, \tag{92}$$

where the curvature K_V of the 2-metric appearing in equation (85), is given by

$$K_V = \frac{2}{V} \, \bar{\eth}_V \eth_V \, V - \frac{2}{V^2} \, \eth_V V \, \bar{\eth}_V V + V^2, \tag{93}$$

Particle model in the null gauge XI

the leading order behavior Ψ_2^0 of the second component of the Weyl tensor is $\Psi_2^0 = -M_0$ and

$$X^0 = 1, \quad X^2 = 0, \quad X^3 = 0; \tag{94}$$

and where $\eth_V$ is the edth operator, in the GHP notation, of the sphere with metric (85). The vacuum Einstein equation can be reduced to a parabolic equation for a scalar depending on three variables, the so called Robinson-Trautman equation; which in our notation has the form

$$-3\,M_0\,\dot{V} = V^4\ \eth^2\bar{\eth}^2\ V - V^3\ \eth^2 V\ \bar{\eth}^2 V; \tag{95}$$

where $\eth$ is the edth operator of the unit sphere. We refer to a line element with V satisfying this equation as a Robinson-Trautman solution. On the other hand, if the RT equation is not required, then the solution is no longer vacuum and there is only one component of the Ricci tensor different from zero, given by[DMG96]

$$\Phi_{22}^{(RT)} = \frac{-3\,M_0\,\frac{\dot{V}}{V}\ - V^3\ \eth^2\bar{\eth}^2\ V + V^2\ \eth^2 V\ \bar{\eth}^2 V}{r^2}; \tag{96}$$

where the *(RT)* is to emphasize the fact that in this case we are using the null tetrad adapted to the null congruence. We refer to this as a Robinson-Trautman geometry.

It is also very interesting to calculate[DMG96] the time variation of the total momentum in these geometries. With respect to the instantaneous Bondi time one has

$$\frac{dP^\alpha}{du_B} = -\frac{1}{4\pi}\int\left(\frac{\partial\sigma^0}{\partial u_B}\frac{\partial\bar{\sigma}^0}{\partial u_B} - \Phi_{22}^0\right)\hat{\ell}^\alpha dS^2; \tag{97}$$

while the time derivative of the total momentum with respect to the Robinson-Trautman time is

$$\frac{dP^\alpha}{du} = -\frac{1}{4\pi}\int\left(\frac{\eth^2 V\bar{\eth}^2 V}{V} - \frac{\Phi_{22}^{(RT)0}}{V^3}\right)\hat{\ell}^\alpha dS^2. \tag{98}$$

It is also convenient to recall the relations

$$\frac{\partial\sigma^0}{\partial u_B} = \frac{\eth^2 V}{V}, \tag{99}$$

$$\Phi_{22}^0 = \frac{\Phi_{22}^{(RT)0}}{V^4}, \tag{100}$$

and

$$\frac{\partial u_B}{\partial u} = V. \tag{101}$$

Particle model in the null gauge XIII

When viewing monopole particles as Robinson-Trautman geometries the time parameter u coincides with the proper time of the particle. Then, the balanced equation of motion can be directly read from equation (97) or (98).

Using the proper time u, one sees that demanding that $\frac{\Phi_{22}^{(RT)0}}{V^3}$ has no $l = 0$ or $l = 1$ spherical harmonic components, provides the appropriate balanced equation of motion; namely

$$\boxed{\frac{dP^\alpha}{du} = -\frac{1}{4\pi}\int \frac{\eth^2 V \bar{\eth}^2 V}{V} \hat{\ell}^\alpha dS^2}. \tag{102}$$

This becomes a rather simple equation that is manageable from the point of view of a numerical calculation.

Particle model in the null gauge XIV

Properties of the particle model in the null gauge

- In studying the particle model from the point of view of the Robinson-Trautman geometries, we have found that the calculation of the equations of motion is direct.
- However in this model, the representation of the spacetime geometry is a little more involved. Although for the calculation of the dynamics, one does not need the complete knowledge of the spacetime geometry.
- In a sense these are the opposite qualities from the previous model based on the harmonic gauge.
- Nevertheless we should remark, that the null gauge for a particle, on a flat background, provides with a direct connection between the local fields and the concept of gravitational radiation. More concretely, we have found that the balanced equation of motion is obtained from the equations

$$\int \frac{\Phi_{22}^{(RT)0}}{V^3}\, \hat{\ell}^{\alpha} dS^2 = 0 \,. \tag{103}$$

This equation can then be demanded for the general case of a non-flat background, as precisely the condition that is needed for the balanced equations of motion.

Particle model in the null gauge XV

Numeric implementation

The numeric implementation for this method of approximation involves the need to solve two equations of motion for the particles that depend on the retarded data. That is we deal with two dynamical times, for the binary system; as opposed to the absolute Newtonian time of post-Newtonian approaches.

To solve a dynamical system taking into account retarded data effects, is already a challenge. We have not found literature on previous work on this field.

To develop the code we have first tackled the toy problem of two particles that interact through Newtonian retarded forces, with relativistic initial conditions.

We will present numerical calculations of our method of approximation in the near future.

Content

1. Introduction
   - Particles as isolated objects
   - Particles as point like objects in special relativity
   - The case of a single spherically symmetric object in general relativity
   - On the nature of other previous approaches
   - Motivation for so much work on the motion of 'particles' in general relativity
2. Previous approaches to the notion of particles by other authors
   - Post-Newtonian approaches
   - Methods of self-force
3. Our previous approach based on the harmonic gauge
   - The decomposition of the metric
   - Auxiliary functions or gauge vector
   - The field equations in relaxed covariant form
   - Iterative approximation method
   - The gravitational field from one particle in the first iteration
4. Particle model in the null gauge
   - Linearized gravity
   - General asymptotically flat spacetimes
   - Monopole particles and Robinson-Trautman geometries
   - Properties of the particle model in the null gauge
   - Numeric implementation
5. Final comments

Final comments I

- When embarking on the construction of a model for compact objects within the particle paradigm in general relativity, it is natural to recur to approximation schemes in terms of some order parameter. One knows from the start that such construction will have sense only if it is thought in terms of finite orders.
- From the characteristics of our model we expect to improve on the range of possible systems that we can study with respect to the ranges covered by the post-Newtonian and the self-force techniques.

Final comments II

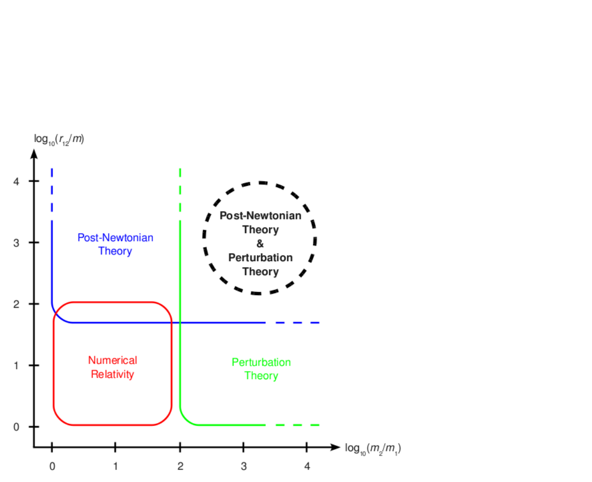


Figure: Graph borrowed from Blanchet presentation.

Final comments III

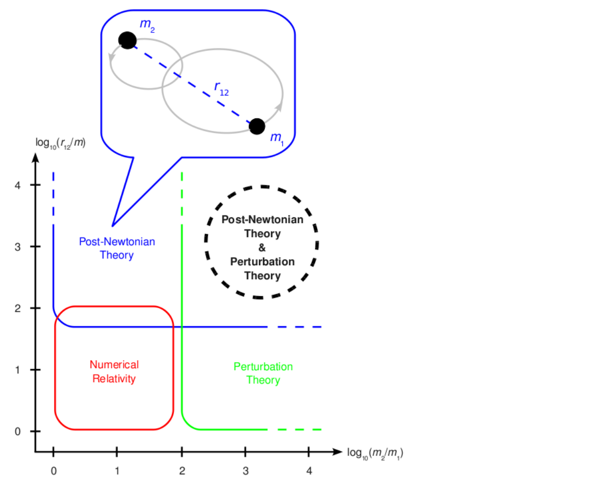


Figure: Graph borrowed from Blanchet presentation.

Final comments IV

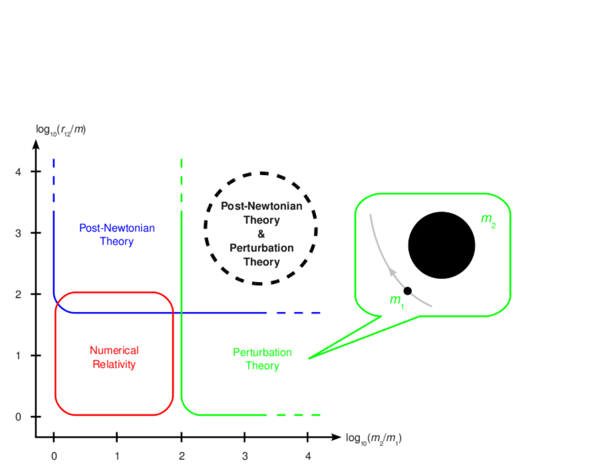


Figure: Graph borrowed from Blanchet presentation.

Final comments V

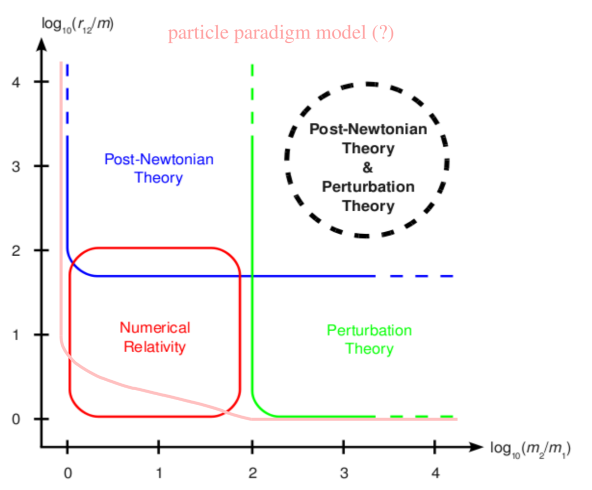


Figure: Our model embraces both other models: so we depict here our expectations for the range of applicability of the model.

Final comments VI

- We plan to improve on this model by making further corrections in terms of the local structure of the field equations.
- We will compare numeric calculations coming from our model with the other approaches to the particle paradigm.
- The main goal is to apply our model to a binary system.

Bib I

B. P. Abbott et al.
Search for gravitational waves from low mass compact binary coalescence in 186 days of LIGO's fifth science run.
Phys. Rev., D80:047101, 2009.

J. Abadie et al.
Search for gravitational waves from compact binary coalescence in LIGO and Virgo data from S5 and VSR1.
Phys. Rev., D82:102001, 2010.

J. Abadie et al.
Search for gravitational waves from binary black hole inspiral, merger and ringdown.
2011.

P. Anninos and S. Brandt.
Head-on collision of two unequal mass black holes.
Phys. Rev. Lett., 81:508, 1998.

Bib II

P. Anninos, D. Hobill, E. Seidel, L. Smarr, and Wai-Mo Suen.
Collision of two black holes.
Phys. Rev. Lett., 71:2851, 1993.

James L. Anderson.
Satisfaction of deDonder and Trautman conditions by radiative solutions of the Einstein field equations.
Gen. Rel. Grav., 4:289–297, 1973.

Luc Blanchet, Guillaume Faye, and Benedicte Ponsot.
Gravitational field and equations of motion of compact binaries to 5/2 post-Newtonian order.
Phys. Rev., D58:124002, 1998.

Sergio Dain and Osvaldo M. Moreschi.
General existence proof for rest frame system in asymptotically flat space-time.
Class. Quantum Grav., 17:3663–3672, 2000.

Bib III

Sergio Dain, Osvaldo M. Moreschi, and Reinaldo J. Gleiser.
Photon rockets and the Robinson-Trautman geometries.
Class. Quantum Grav., 13(5):1155–1160, 1996.

Albert Einstein, L. Infeld, and B. Hoffmann.
The Gravitational equations and the problem of motion.
Annals Math., 39:65–100, 1938.

H. Friedrich.
On the hyperbolicity of einstein's and other gauge field equations.
Commun. Math. Phys., 100:525–543, 1985.

R. Geroch, A. Held, and R. Penrose.
A space-time calculus based on pairs of null directions.
J. Math. Phys., 14:874–881, 1973.

Samuel E. Gralla, Abraham I. Harte, and Robert M. Wald.
A Rigorous Derivation of Electromagnetic Self-force.
Phys.Rev., D80:024031, 2009.

Bib IV

Emanuel Gallo and Osvaldo M. Moreschi.
Approximation Method for the Relaxed Covariant Form of the Gravitational Field Equations for Particles.
Journal of Modern Physics, 3:1247–1254, 2012b.

Emanuel Gallo and Osvaldo M. Moreschi.
New derivation for the equations of motion for particles in electromagnetism.
Phys.Rev., D85:065005, 2012.

Walter D. Goldberger and Ira Z. Rothstein.
Effective field theory of gravity for extended objects.
Phys.Rev.D, 73:104029(22), 2006.

Peter Havas and Joshua N. Goldberg.
Lorentz-Invariant Equations of Motion of Point Masses in the General Theory of Relativity.
Phys. Rev., 128:398–414, 1962.

Bib V

Osvaldo M. Moreschi and Sergio Dain.
Estimates of the total gravitational radiation in the head-on black hole collision.
Phys. Rev. D, 53(4):R1745–R1749, 1996.
Rapid Communication.

Osvaldo M. Moreschi and Sergio Dain.
Rest frame system for asymptotically flat space-times.
J. Math. Phys., 39(12):6631–6650, 1998.

Osvaldo M. Moreschi.
On angular momentum at future null infinity.
Class. Quantum Grav., 3:503–525, 1986.

Osvaldo M. Moreschi.
Supercenter of mass system at future null infinity.
Class. Quantum Grav., 5:423–435, 1988.

Bib VI

Osvaldo M. Moreschi.
Total energy radiated in the head-on black hole collision with arbitrary mass ratio.
Phys. Rev. D, 59:084018, 1999.

Osvaldo M. Moreschi.
Intrinsic angular momentum and center of mass in general relativity.
Class.Quantum Grav., 21:5409–5425, 2004.

Eric Poisson.
The motion of point particles in curved spacetime.
Living Rev. Rel., 6:136, (2004), http://www.livingreviews.org/lrr-2004-6.

Eric Poisson, Adam Pound, and Ian Vega.
The Motion of point particles in curved spacetime.
Living Rev.Rel., 14:7, 2011.

I. Robinson and R. Trautman.
Some spherical gravitational waves in general relativity.
Proc. R. Soc. A, 265:463–473, 1962.

Bib VII

R. Sachs.
Asymptotic symmetries in gravitational theory.
Phys. Rev., 128:2851–2864, 1962.

Gerhard Schäfer.
Post-Newtonian methods: Analytic results on the binary problem.
arXiv:0910.2857v1[gr-qc], 2009.

M. Walker and C. M. Will.
The approximation of radiative effects in relativistic gravity - Gravitational radiation reaction and energy loss in nearly Newtonian systems.
Astrophys. J.(Letters), 242:L129–L133, December 1980.