Black Holes Shadows in Konoplya-Stuchlik-Zhidenko metrics

Gastón Briozzo[1] and Emanuel Gallo [1,2]

[1]Facultad de Matemática, Astronomía, Física y Computación-UNC-Córdoba-Argentina

[2]IFEG-CONICET

Employing the family of stationary, axisymmetric and asymptotically flat metrics proposed in [Konoplya, Stuchlik & Zhidenko (2018)], which allows the separation of variables in the Klein-Gordon and Hamilton-Jacobi equations, we imitated the work [Perlick & Tsupko (2017)] to analytically obtain the shadow contour curve of rotating black holes immersed in plasmatic environments. This procedure was performed considering different Kerr-like spacetime models and different plasma concentrations. Finally, we imitated the work [Grenzebach (2015)] to study the effect of relativistic aberration on the shadow of black holes.

The **KSZ metrics** are expressed in contravariant form as

$$[g^{\mu\nu}] = \frac{1}{\rho^2}\begin{pmatrix} a^2\sin^2\theta - \frac{\left(r^2R_\Sigma + a^2\right)^2}{\Delta} & 0 & 0 & -\frac{arR_M}{\Delta} \\ 0 & \frac{\Delta}{R_B^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{arR_M}{\Delta} & 0 & 0 & \frac{1}{\sin^2\theta} - \frac{a^2}{\Delta} \end{pmatrix},$$

where $R_\Sigma(r)$, $R_M(r)$ and $R_B(r)$ are functions only of r that satisfy the limits

$$\lim_{r\to\infty} R_B(r) = 1,\ \lim_{r\to\infty} R_\Sigma(r) = 1,\ \lim_{r\to\infty} \frac{R_M(r)}{r} = 0,$$

and we have introduced the following functions

$$\Delta(r) = r^2R_\Sigma(r) - rR_M(r) + a^2, \quad \rho^2(r,\theta) = r^2R_\Sigma(r) + a^2\cos^2\theta.$$

By these expressions, we can se that the **Hamilton-Jacobi equation**,

$$\frac{1}{2}\left[g^{\mu\nu}(\vec{x})\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \omega_p^2(\vec{x})\right] = 0,$$

will be separable only if we consider **plasma environments with frequency** $\omega_p^2(r,\theta) = \frac{f_r(r)+f_\theta(\theta)}{\rho^2}$.

We then write the Hamilton-Jacobi equation,

$$H = \frac{1}{2}\frac{1}{\rho^2}\left[-\frac{\left((r^2R_\Sigma + a^2)p_t + ap_\phi\right)^2}{\Delta} + \frac{\Delta}{R_B^2}\left(\frac{\partial S}{\partial r}\right)^2 + \left(a\sin\theta + \frac{1}{\sin\theta}p_\phi\right)^2 + \left(\frac{\partial S}{\partial\theta}\right)^2 + \rho^2\omega_p^2\right],$$

and define the **generalized Carter constant**,

$$K := \frac{\left((r^2R_\Sigma(r) + a^2)p_t + ap_\phi\right)^2}{\Delta(r)} - \frac{\Delta(r)}{R_B^2(r)}\left(\frac{\partial S}{\partial r}\right)^2 - f_r(r)$$

$$K := \left(a\sin\theta + \frac{1}{\sin\theta}p_\phi\right)^2 + \left(\frac{\partial S}{\partial\theta}\right)^2 + f_\theta(\theta).$$

From this, we can express the momentum as follows

$$p_t = \omega_\infty,$$

$$p_r^2 = \frac{R_B^2(r)}{\Delta(r)}\left[-K + \frac{\left((r^2R_\Sigma(r) + a^2)p_t + ap_\phi\right)^2}{\Delta(r)} - f_r(r)\right],$$

$$p_\theta^2 = K - \left(a\sin\theta + \frac{1}{\sin\theta}p_\phi\right)^2 - f_\theta(\theta),$$

$$p_\phi = \omega_\infty b n_0.$$

From Hamilton's equations,

$$\frac{dx^\alpha}{d\lambda} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{d\lambda} = -\frac{\partial H}{\partial x^\alpha}, \quad H(x^\alpha, p_\alpha) = 0,$$

we obtains the **first-order equations of motion** for the photon,

$$\dot{t} = \frac{1}{\rho^2}\left[\left(a^2\sin^2\theta - \frac{\left(r^2R_\Sigma + a^2\right)^2}{\Delta}\right)p_t - \frac{arR_M}{\Delta}p_\phi\right],$$
$$\dot{r}^2 = \frac{1}{\rho^4}\frac{-K\Delta + \left((r^2R_\Sigma + a^2)p_t + ap_\phi\right)^2 - \Delta f_r}{R_B^2},$$
$$\dot{\theta}^2 = \frac{1}{\rho^4}\left[K - \left(a\sin\theta\, p_t + \frac{1}{\sin\theta}p_\phi\right)^2 - f_\theta\right],$$
$$\dot{\phi} = \frac{1}{\rho^2}\left[\left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)p_\phi - \frac{arR_M}{\Delta}p_t\right].$$

The **photon región** is given by $\dot{r} = 0, \ddot{r} = 0$, resulting in

$$0 = -(K + f_r)\Delta + \left((r^2R_\Sigma + a^2)p_t + ap_\phi\right)^2,$$

$$0 = -(K + f_r)\Delta' - f_r{}'\Delta + 4rR_\Sigma\left((r^2R_\Sigma + a^2)p_t + ap_\phi\right)p_t,$$

So that the **constants of motion** can be expressed as

$$K(r_p) = \Delta\left(\frac{2r\ \ {}_\Sigma p_t}{\Delta'}\right)^2\left[1 + \sqrt{1 - \left(\frac{\Delta'}{2rR_\Sigma p_t}\right)^2\frac{f_r{}'}{\Delta'}}\right]^2 - f_r,$$

$$p_\phi(r_p) = 2\frac{\Delta}{\Delta'}\frac{rR_\Sigma p_t}{a}\left[1 + \sqrt{1 - \left(\frac{\Delta'}{2rR_\Sigma p_t}\right)^2\frac{f_r{}'}{\Delta'}}\right] - \left(\frac{r^2R_\Sigma}{a} + a\right)p_t.$$

The **boundaries** of the photon región are obtained by solving $\dot{\theta}(r_p) = 0$,

$$\sin^2\theta\,(K - f_\theta)_{r_{p,min/max}} = \left(p_\phi - a\sin^2\theta\, p_t\right)^2_{r_{p,min/max}}.$$

We propose the **orthonormal tetrad**

$$e_0 = \frac{r^2R_\Sigma + a^2}{\rho\sqrt{\Delta}}\partial_t + \frac{a}{\rho\sqrt{\Delta}}\partial_\phi, \qquad e_1 = \frac{1}{\rho}\partial_\theta,$$
$$e_2 = -\frac{1}{\rho\sin\theta}\partial_\phi - \frac{a\sin\theta}{\rho}\partial_t, \qquad e_3 = -\frac{\sqrt{\Delta}}{\rho R_B}\partial_r$$

And write the tangent vector to the photon at the observer's position (r_O, ϑ_O)

$$\dot{\lambda} = \dot{t}\partial_t + \dot{r}\partial_r + \dot{\theta}\partial_\theta + \dot{\phi}\partial_\phi,$$
$$\dot{\lambda} = -\alpha e_0 + \beta(\sin\Theta\cos\Phi\, e_1 + \sin\Theta\sin\Phi\, e_2 + \cos\Theta\, e_3),$$

Where we have defined the celestial coordinates of the observer, Θ y Φ.

Given $g(\dot{\lambda}, \dot{\lambda}) = -\alpha^2 + \beta^2 = -\omega_p^2(r_O, \vartheta_O)$ y $\alpha = g(\dot{\lambda}, e_0)$, we have

$$\alpha = \left.\frac{(r^2R_\Sigma + a^2)p_t + ap_\phi}{\rho\sqrt{\Delta}}\right|_{(r_O, \vartheta_O)},$$

$$\beta = \left.\sqrt{\frac{\left((r^2R_\Sigma + a^2)p_t + ap_\phi\right)^2}{\rho^2\Delta} - \omega_p^2}\right|_{(r_O, \vartheta_O)}.$$

Comparing both expressions for $\dot{\lambda}$ and equating the terms with the same partial derivatives, we obtain

$$\sin\Theta = \sqrt{\frac{\Delta(r_O)\left[K(r_p) - f_\theta(\vartheta_O)\right]}{\left[(r_O^2R_\Sigma(r_O) + a^2)p_t + ap_\phi(r_p)\right]^2 - \Delta(r_O)\left[f_r(r_O) + f_\theta(\vartheta_O)\right]}},$$

$$\sin\Phi = -\frac{p_\phi(r_p) + ap_t\sin^2\vartheta_O}{\sin\vartheta_O\sqrt{K(r_p) - f_\theta(\vartheta_O)}}.$$

To plot the shadow, we will use **stereographic projections**

$$X(r_p) = -2\tan\left[\frac{\Theta(r_p)}{2}\right]\sin\left[\Phi(r_p)\right],$$
$$Y(r_p) = -2\tan\left[\frac{\Theta(r_p)}{2}\right]\cos\left[\Phi(r_p)\right].$$

We will consider the following **spacetime models**

Metric	$R_B(r)$	$R_M(r)$	$R_\Sigma(r)$	Horizon
Kerr	1	$2M$	1	$r_h = M \pm \sqrt{M^2 - a^2}$
Kerr-Newman	1	$2M - \frac{Q^2}{r}$	1	$r_h = M \pm \sqrt{M^2 - a^2 - Q^2}$
Kerr Modified	1	$2M + \frac{\eta}{r^2}$	1	$0 = r_h^2 - 2Mr_h - \frac{\eta}{r_h} + a^2$
Kerr-Sen	1	$2M$	$1 + \frac{2b}{r}$	$r_h = M - b \pm \sqrt{(M-b)^2 - a^2}$
Braneworld	1	$2M + \frac{Q^2}{r}$	1	$r_h = M \pm \sqrt{M^2 - a^2 + Q^2}$
Dilaton	1	$2M - \frac{q + r_0^2}{r}$	$1 - \frac{r_0^2}{r^2}$	$r_h = M \pm \sqrt{M^2 - a^2 - q}$

We will call Q to each metric parameter $(\eta, b, r_0\ (q = 2r_0))$ and respect the following constraints

Metric	K-N	K M	K-S	BW	Di
Q_{max}	1	0,2	0,5	1	0,5
a_{max}	$\sqrt{1-Q^2}$	1	$1-Q$	$\sqrt{1+Q^2}$	$\sqrt{1-2Q}$

We will use the following **plasma electronic frequencies**

$$\omega_{p,1}^2(r,\theta) = 0, \quad \omega_{p,2}^2(r,\theta) = \frac{\omega_c^2 m^2 (1 + 2\sin^2\theta)}{r^2 R_\Sigma(r) + a^2\cos^2\theta}, \quad \omega_{p,3}^2(r,\theta) = \frac{\omega_c^2 \sqrt{m^3 r}}{r^2 R_\Sigma(r) + a^2\cos^2\theta}$$

$$\omega_{p,4}^2(r,\theta) = \omega_c^2, \quad \omega_{p,5}^2(r,\theta) = \frac{\omega_c^2 m^3}{r^3 R_\Sigma(r) + r a^2\cos^2\theta}, \quad \omega_{p,6}^2(r,\theta) = \frac{\omega_c^2 m^2 exp[(r_1 - r)/r_0]}{r^2 R_\Sigma(r) + a^2\cos^2\theta}$$

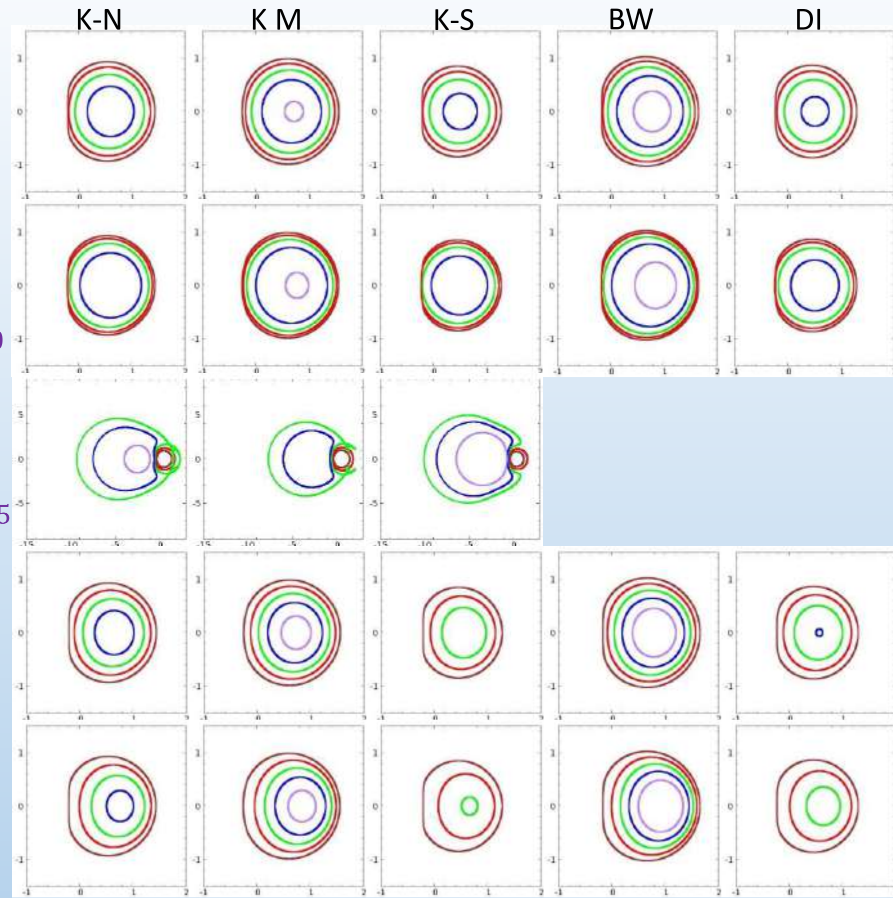
Profile 2
$\frac{\omega_c^2}{\omega_\infty^2} = 0{,}00\ 2{,}25\ 4{,}50\ 6{,}75\ 8{,}90$

Profile 3
$\frac{\omega_c^2}{\omega_\infty^2} = 0{,}00\ 3{,}75\ 7{,}50\ 11{,}25\ 15{,}00$

Profile 4
$\frac{\omega_c^2}{\omega_\infty^2} = 0{,}00\ 0{,}80\ 1{,}085\ 1{,}20\ 1{,}345$

Profile 5
$\frac{\omega_c^2}{\omega_\infty^2} = 0{,}0\ 17{,}5\ 35{,}0\ 52{,}5\ 70{,}0$

Profile 6
$\frac{\omega_c^2}{\omega_\infty^2} = 0{,}0\ 32{,}5\ 65{,}0\ 97{,}5\ 130{,}0$



$$\frac{a}{a_{max}} = 0{,}999, \qquad r_O = 5, \qquad \vartheta_O = \frac{\pi}{2}, \qquad \frac{Q}{Q_{max}} = \sqrt{0{,}25}$$

As the angular momentum $\boldsymbol{a}$ of the black hole increases, the shadow shifts to the right, shrinks slightly in size, and adopts a characteristic D-shape. As the plasma density increases, the shadow shrinks and loses the D-shape, appearing a cutoff frequency $\omega_c^2/\omega_\infty^2$ beyond which shadowing ceases. The value of that frequency depends on both the metric model and the plasma distribution. In profile 4 we see "**fish-tails**", frequencies at which the contour curve happens to enclose the illuminated area of the sky, now surrounded by the shadow.

So far, we have considered an observer positioned at coordinates (r_O, ϑ_O) whose 4-velocity is given by the $e_0$component of the orthonormal tetrad. This observer is called the standard observer (s.o.). For another observer with a 3-velocity (v_1, v_2, v_3) with respect to the s.o. the orthonormal tetrad will be different and the shape of the contour curve of the black hole shadow will be affected due to the effect of relativistic aberration.

The orthonormal tetrad of the standard observer can be expressed as

$$[e_i^\mu] = \begin{pmatrix} \frac{r^2 R_\Sigma + a^2}{\rho\sqrt{\Delta}} & 0 & 0 & \frac{a}{r\rho\sqrt{\Delta}} \\ 0 & 0 & \frac{1}{\rho} & 0 \\ -\frac{a\sin\theta}{\rho} & 0 & 0 & -\frac{1}{\rho\sin\theta} \\ 0 & -\frac{\sqrt{\Delta}}{\rho R_B} & 0 & 0 \end{pmatrix},$$

An observer with 3-velocity (v_1, v_2, v_3) respect to the s.o. will have the tetrad

$$\tilde{e}_0 = \frac{e_0 + v_1 e_1 + v_2 e_2 + v_3 e_3}{\sqrt{1-v^2}}, \quad \tilde{e}_1 = \frac{(1-v_2^2)e_1 + v_1(e_0 + v_2 e_2)}{\sqrt{1-v_2^2}\sqrt{1-v_1^2-v_2^2}},$$

$$\tilde{e}_2 = \frac{e_2 + v_2 e_0}{\sqrt{1-v_2^2}}, \quad \tilde{e}_3 = \frac{(1-v_1^2-v_2^2)e_3 + v_3(e_0 + v_1 e_1 + v_2 e_2)}{\sqrt{1-v^2}\sqrt{1-v_1^2-v_2^2}}.$$

We can express these coefficients, $\tilde{e}_i = k_i^\mu \partial_\mu$, in terms of the matrix k

$$k_i^\mu = \begin{pmatrix} \frac{e_0^t + v_2 e_2^t}{\sqrt{1-v^2}} & \frac{v_3 e_3^r}{\sqrt{1-v^2}} & \frac{v_1 e_1^\theta}{\sqrt{1-v^2}} & \frac{e_0^\phi + v_2 e_2^\phi}{\sqrt{1-v^2}} \\ \frac{v_1(e_0^t + v_2 e_2^t)}{\sqrt{1-v_2^2}\sqrt{1-v_1^2-v_2^2}} & 0 & \sqrt{\frac{1-v_2^2}{1-v_1^2-v_2^2}}\, e_1^\theta & \frac{v_1\left(e_0^\phi + v_2 e_2^\phi\right)}{\sqrt{1-v_2^2}\sqrt{1-v_1^2-v_2^2}} \\ \frac{v_2 e_0^t + e_2^t}{\sqrt{1-v_2^2}} & 0 & 0 & \frac{v_2 e_0^\phi + e_2^\phi}{\sqrt{1-v_2^2}} \\ \frac{v_3(e_0^t + v_2 e_2^t)}{\sqrt{1-v^2}\sqrt{1-v_1^2-v_2^2}} & \sqrt{\frac{1-v_1^2-v_2^2}{1-v^2}}\, e_3^r & \frac{v_3 v_1 e_1^\theta}{\sqrt{1-v^2}\sqrt{1-v_1^2-v_2^2}} & \frac{v_3\left(e_0^\phi + v_2 e_2^\phi\right)}{\sqrt{1-v^2}\sqrt{1-v_1^2-v_2^2}} \end{pmatrix}.$$

We write the tangent vector to the photon

$$\dot{\lambda} = \dot{t}\partial t + \dot{r}\partial r + \dot{\theta}\partial\theta + \dot{\phi}\partial\phi,$$
$$\dot{\lambda} = -\alpha\tilde{e}_0 + \beta(\sin\Theta\cos\Phi\,\tilde{e}_1 + \sin\Theta\sin\Phi\,\tilde{e}_2 + \cos\Theta\,\tilde{e}_3),$$

where α and β are now expressed as

$$\alpha = g\left(\dot{\lambda}, \tilde{e}_0\right) = k_0^t p_t + k_0^r p_r + k_0^\theta p_\theta + k_0^\phi p_\phi,$$
$$\beta = \sqrt{\left(k_0^t p_t + k_0^r p_r + k_0^\theta p_\theta + k_0^\phi p_\phi\right)^2 - \omega_p^2}.$$

By comparing both expressions for $\dot{\lambda}$ and equating the terms with same partial derivatives, we obtain the following set of coupled equations

$$\dot{t} = -\alpha k_0^t + \beta k_1^t \sin\Theta\cos\Phi + \beta k_2^t \sin\Theta\sin\Phi + \beta k_3^t \cos\Theta,$$
$$\dot{r} = -\alpha k_0^r + \beta k_3^r \cos\Theta,$$
$$\dot{\theta} = -\alpha k_0^\theta + \beta k_1^\theta \sin\Theta\cos\Phi + \beta k_3^\theta \cos\Theta,$$
$$\dot{\phi} = -\alpha k_0^\phi + \beta k_1^\phi \sin\Theta\cos\Phi + \beta k_2^\phi \sin\Theta\sin\Phi + \beta k_3^\phi \cos\Theta,$$

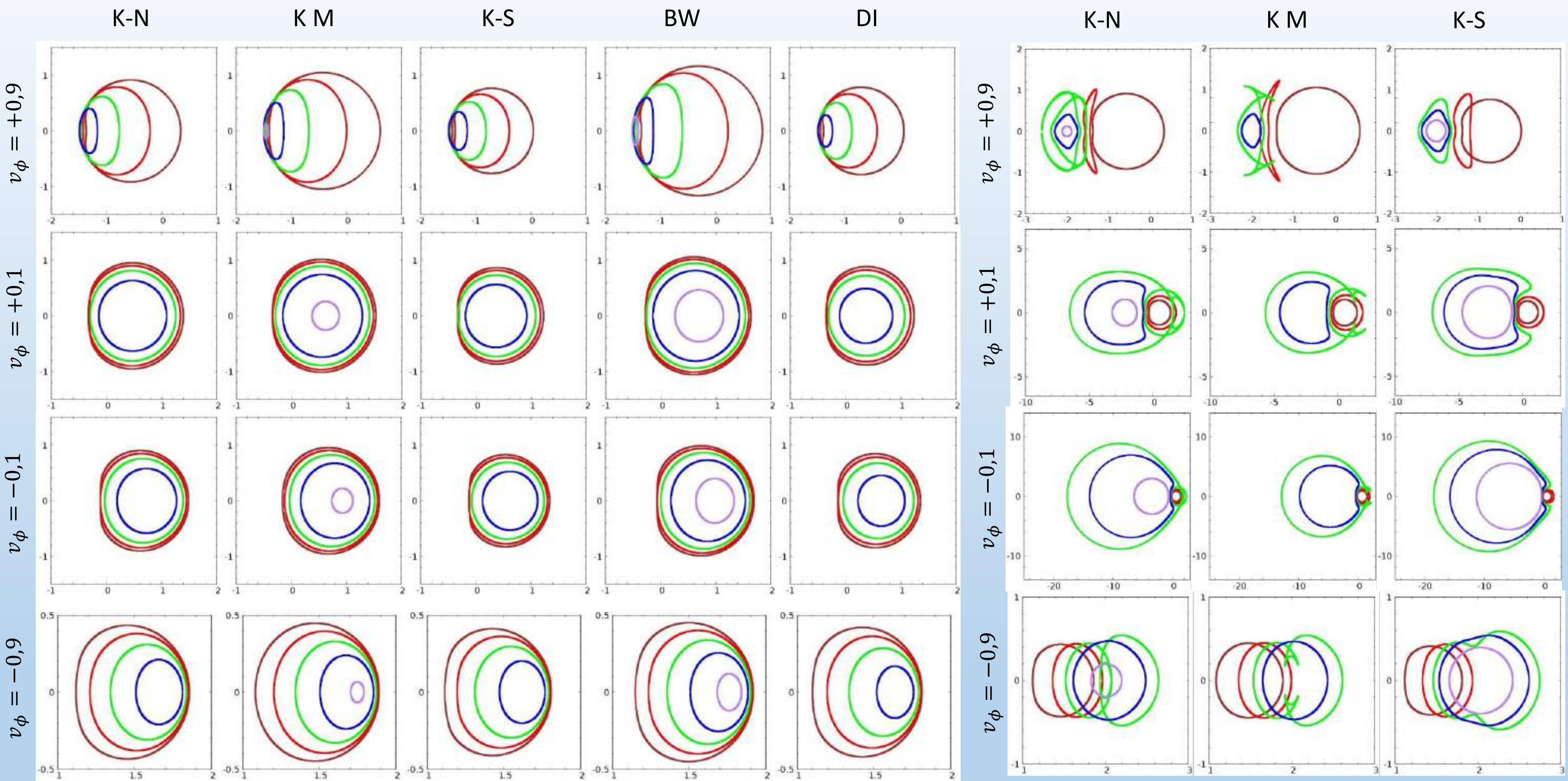
from which we find the following expressions

$$\cos\Theta = \frac{\dot{r} + \alpha k_0^r}{\beta k_3^r},$$

$$\sin\Phi = \frac{\dot{\phi} + \alpha k_0^\phi - (\dot{\theta} + \alpha k_0^\theta)\frac{k_1^\phi}{k_1^\theta} + \beta\frac{\dot{r} + \alpha k_0^r}{\beta k_3^r}\left(k_3^\theta \frac{k_1^\phi}{k_1^\theta} - k_3^\phi\right)}{\beta k_2^\phi \sqrt{1 - \left(\frac{\dot{r} + \alpha k_0^r}{\beta k_3^r}\right)^2}}.$$

Profile 3, $\frac{\omega_c^2}{\omega_\infty^2}$ = 0,00 3,75 7,50 11,25 15,00

Profile 4, $\frac{\omega_c^2}{\omega_\infty^2}$ = 0,000 0,800 1,085 1,200 1,345



$$\frac{a}{a_{max}} = 0,999$$
$$r_O = 5$$
$$\vartheta_O = \frac{\pi}{2}$$
$$\frac{Q}{Q_{max}} = \sqrt{0,25}$$

We see that the aberration deforms and shifts the shadow depending on the frequency. For profile 4 a **critical transition velocity** appears, $v_c \approx -0,1$, for which the length of the shadow contour curve diverges and the shadow becomes half of the observer's sky. At lower velocities, the curve closes to the right of the black hole, recovering a shadow enclosed by an illuminated sky.